# Parallel Derivation of Efficient Continuous/Discrete Explicit Runge-Kutta Methods

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**Abstract**

New discrete and continuous explicit Runge-Kutta methods are developed which employ the FSAL (first-same-as-last) technique and other than possibly the FSAL evaluation no additional evaluations are required for the continuous methods. Efficiency is measured from a practical point of view by consideration of cost (function evaluations) against global accuracy. The free parameters which are associated with the new methods are chosen to satisfy this efficiency criterion. Numerical tests are carried out on a number of standard problems and these show that the new higher order methods offer significant practical advantages.

## 1 Introduction

This work has been largely influenced by the suggestion made (page 194) in [9] that new methods should be looked for which use all stages associated with the discrete and continuous processes.

Consideration is given to the first order system of non-stiff initial value ordinary differential equations

$$\boldsymbol{y}'(x) = \boldsymbol{f}[x, \boldsymbol{y}(x)], \quad \boldsymbol{y}(x_0) \text{ known},$$

which may be solved using a Runge-Kutta (RK) triple [4], denoted by RKT$q(p)q^*$, of the form:

$$
\begin{aligned}
\widehat{\boldsymbol{y}}_{n+1} &= \widehat{\boldsymbol{y}}_n + h_n \sum_{i=1}^{s} \widehat{b}_i \boldsymbol{g}_i \\
\boldsymbol{y}_{n+1} &= \widehat{\boldsymbol{y}}_n + h_n \sum_{i=1}^{s^*} b_i \boldsymbol{g}_i \\
\boldsymbol{y}^*_{n+\sigma} &= \widehat{\boldsymbol{y}}_n + \sigma h_n \sum_{i=1}^{s^*} b^*_i(\sigma) \boldsymbol{g}_i
\end{aligned} \tag{1}
$$

where $\boldsymbol{g}_i = \boldsymbol{f}[x_n + c_i h_n,\ \widehat{\boldsymbol{y}}_n + h_n \sum_{j=1}^{i-1} a_{ij} \boldsymbol{g}_j], \quad i = 1, 2, \ldots, s^*,$ and in which $x_{n+1} = x_n + h_n,\ h_n = \theta(x_n)h,\ 0 \leq \theta(x_n) \leq 1,\ \widehat{\boldsymbol{y}}_0 = \boldsymbol{y}(x_0),\ x_{n+\sigma} = x_n + \sigma h_n$, the discrete embedded pair are of orders $q$ and $p$ $(q > p)$, the dense formula is of order $q^*$ and it is assumed that $\boldsymbol{g}_{s^*}$ is the FSAL evaluation so that $\boldsymbol{y}^*_{n+\sigma}$ is $C^1$ on $[x_n, x_{n+1}]$. Local extrapolation, in which the integration is propagated from the higher order approximation $\widehat{\boldsymbol{y}}_n$ to the true solution $\boldsymbol{y}(x_n)$, is the preferred mode of operation [3] and is therefore assumed. Thus the lower order discrete process is only used to control the step size. The FSAL assumption requires

$$
c_{s^*} = 1, \quad \widehat{b}_{s^*} = 0, \quad a_{s^*j} = \widehat{b}_j, \quad j = 1, 2, ..., s = s^* - 1. \tag{2}
$$

Under appropriate conditions, the local truncation errors $\widehat{\boldsymbol{t}}_{n+1}$ and $\boldsymbol{t}_{n+1}$, at $x = x_{n+1}$, of the RK$q$ and RK$p$ processes and the local truncation error $\boldsymbol{t}_{n+\sigma}$ of the continuous RK$q^*$ process at $x_{n+\sigma}$ may be written [1]

$$
\widehat{\boldsymbol{t}}_{n+1} = \sum_{i=q+1}^{\infty} h_n^i \sum_{j=1}^{n_i} \widehat{\tau}_j^{(i)} \boldsymbol{F}_j^{(i)}[x_n, \boldsymbol{y}(x_n)],
$$

$$
\boldsymbol{t}_{n+1} = \sum_{i=p+1}^{\infty} h_n^i \sum_{j=1}^{n_i} \tau_j^{(i)} \boldsymbol{F}_j^{(i)}[x_n, \boldsymbol{y}(x_n)],
$$

$$
\boldsymbol{t}_{n+\sigma} = \sum_{i=q^*+1}^{\infty} (\sigma h_n)^i \sum_{j=1}^{n_i} \tau_j^{*(i)} \boldsymbol{F}_j^{(i)}[x_n, \boldsymbol{y}(x_n)],
$$

where the $\boldsymbol{F}_j^{(i)}$ are the elementary differentials of $\boldsymbol{f}$ and the $\widehat{\tau}_j^{(i)}, \tau_j^{(i)}$ and $\tau_j^{*(i)}$ are truncation error coefficients, which are dependent on the parameters

$c_i, a_{ij}, \widehat{b}_i, b_i$ and $b_i{}^*$ and the global errors at $x_n$ and $x_{n+\sigma}$ are $\boldsymbol{\varepsilon}_n = \widehat{\boldsymbol{y}}_n - \boldsymbol{y}(x_n)$ and $\boldsymbol{\varepsilon}_{n+\sigma} = \boldsymbol{y}^*_{n+\sigma} - \boldsymbol{y}(x_{n+\sigma})$ respectively. The ordering of the elementary differentials is that adopted by Dormand and Prince [3] to generate the RK equations of condition. The difference $\widehat{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}$ is an estimate of $\boldsymbol{t}_{n+1}$ and can be used to monitor the step size. Usually $p = q - 1$ but sometimes $p = q - 2$ [5] or a pair of lower order formulae [9] can be used successfully to give "stretched" estimation.

The development of embedded RKq(p) pairs involves a number of free parameters. Prince and Dormand [13] outlined a number of criteria as to how these parameters should best be chosen. The main criterion was to choose those parameters which affected the higher order formula so that the norm of the principal truncation error terms was small since this tended to give better global accuracy. Additionally the RK coefficients should not be very large and they should not lead to a lower order formula which gave poor local error estimation. The initial development of RK embedded pairs was carried out without any reference to continuous processes and this resulted in such processes being "added on" at a later stage to the embedded pair which, except for very low orders, implied that $s^* > s$ and significantly so for high order pairs. The order $(q^*)$ of the continuous process is a significant factor in determining the value of $s^*$. Dormand and Prince [4] point out that it is sufficient for $q^* = q - 1$ to have both the discrete and continuous global errors of $O(h^q)$. However as pointed out in [7] other applications may require $q^* = q$ so that the local error in the continuous process is asymptotically negligible compared to the global error at the discrete points. Frequently, therefore, in the development of continuous processes for known embedded pairs there were additional RK parameters available which only affected the continuous process. The question then arose as to how best to choose these additional parameters. Clearly emphasis needs to be placed on the norm of the principal truncation error coefficients in the continuous process and this has been considered by Calvo et al. [2] who have used the additional free parameters to obtain a small value for

$$I_C = \int_0^1 \|\sigma^{q^*+1}\boldsymbol{\tau}^{*(q^*+1)}\|_2 \; d\sigma \tag{3}$$

However, Baker et al. [1], point out that the choice of these parameters should be made according to how they affect the continuous global error and therefore should be chosen to give a small value for

$$I_B = \int_0^1 \|\sigma^{q^*+1}\boldsymbol{\tau}^{*(q^*+1)} - \sigma\widehat{\boldsymbol{\tau}}^{(q^*+1)}\|_2 \; d\sigma \tag{4}$$

where it should be noted that if $q^* = q - 1$ then $I_C$ and $I_B$ are identical since all the $\hat{\tau}_j^{(q)}$ are zero. Thus when $s^* > s$ and there are additional free parameters which do not affect the main discrete process it seems preferable to devise the processes such that $A^{(q+1)} = \|\hat{\boldsymbol{\tau}}^{(q+1)}\|_2$ is small which tends to give smaller discrete global error and to choose the additional parameters so that $I_B$ is small (while also taking into account the size of the RK coefficients and the affect on the local error estimation process).

Initially this work will assume that $q^* = q$ so that $I_C$ and $I_B$ are not the same and from (1) and (2) there are no additional parameters that do not affect the main discrete process. The question arises therefore as to whether the free parameters should be chosen mainly to give small $A^{(q+1)}$ or whether in addition the size of $I_B$ should be taken into account. From Baker et al. [1] the continuous global error can be expressed as

$$\boldsymbol{\varepsilon}_{n+\sigma} = \boldsymbol{\varepsilon}_n + \boldsymbol{G}_{n+\sigma} + \boldsymbol{t}^*_{n+\sigma} + O(h^{q+2})$$

showing that the continuous global error is directly dependent on that at the discrete points which suggests that the priority should be for small $A^{(q+1)}$.

In considering the various values of $q = q^*$ this work was guided by the entries for $N_q$, $D_q$ and $C_q$ in Table 1 where $N_q$ is the number of RK equations of condition for order q, $D_q$ is the minimum number of stages for an explicit discrete RK process and $C_q$ is the minimum number of stages for an explicit continuous RK process. The values of $C_q,\ q = 1, 2, \ldots, 6$, in Table 1 are

Table 1: Values of $N_q$, $D_q$ and $C_q$ for $q = 1, 2, \ldots, 10$

| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_q$ | 1 | 2 | 4 | 8 | 17 | 37 | 85 | 200 | 486 | 1205 |
| $D_q$ | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 11 | $\leq 15$ | $\leq 16$ |
| $C_q$ | 1 | 2 | 4 | 6 | 8 | 11 | $\leq 14$ | $\leq 18$ | $\leq 22$ | |

those given by Enright et al. [7]. The entries for $C_7$, $C_8$ and $C_9$ are obtained from the analysis in the other sections of this work. Note that the values of $D_9$ and $D_{10}$ are not precisely known. From the work of Hairer [8] it was known that $D_{10} \leq 17$ but this entry can now be replaced by $\leq 16$ since a 16-stage RK10 process is given in Table 2 but it should be noted that all the free parameters have been chosen randomly. The process is only presented in order to show that such a process is possible.

Table 2: A 16-stage explicit RK10 process where $a_{17j} = b_j,\ j = 1, 2, \ldots, 16$, $c_i = \sum_{j=1}^{i-1} a_{ij},\ i = 1, 2, \ldots, 16$ and the table only contains the non-zero $a_{ij}$.

| $i$ | $j$ | $Numerator(a_{ij})$ | $Denominator(a_{ij})$ | $i$ | $j$ | $Numerator(a_{ij})$ | $Denominator(a_{ij})$ |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 9 | 13 | 4 | −144031 | 9768285 |
| 3 | 1 | −5 | 6 | 13 | 5 | 4025343400 | 2690185689 |
| 3 | 2 | 25 | 18 | 13 | 6 | 94166721047 | 23473655658 |
| 4 | 1 | 5 | 24 | 13 | 7 | 7335448531 | 8595710442 |
| 4 | 3 | 5 | 8 | 13 | 8 | −44196265055873 | 56837005275636 |
| 5 | 1 | 121397 | 1318200 | 13 | 9 | −51978654834616 | 10620609181875 |
| 5 | 3 | 25721 | 878800 | 13 | 10 | −211398749870620672 | 226295508183215625 |
| 5 | 4 | −32657 | 2636400 | 13 | 11 | 101535598 | 486253125 |
| 6 | 1 | 121 | 3060 | 13 | 12 | −1082198294528 | 133223113814175 |
| 6 | 4 | 1 | 20340 | 14 | 1 | 2899 | 96 |
| 6 | 5 | 2197 | 17289 | 14 | 2 | 25 | 18 |
| 7 | 1 | 161 | 340 | 14 | 6 | 9022833 | 58667 |
| 7 | 4 | 9 | 565 | 14 | 7 | 2685 | 77 |
| 7 | 5 | −10985 | 5763 | 14 | 8 | −6413500779 | 180382240 |
| 7 | 6 | 23 | 12 | 14 | 9 | −3445903488009 | 24066350000 |
| 8 | 1 | 426932 | 1428595 | 14 | 10 | −50803557531648 | 1236558640625 |
| 8 | 4 | 82872 | 9495955 | 14 | 11 | 17425287 | 1750000 |
| 8 | 5 | −111080320 | 96858741 | 14 | 12 | −255879168 | 687061375 |
| 8 | 6 | 64192 | 50421 | 14 | 13 | −4201227 | 444080 |
| 8 | 7 | 2312 | 16807 | 15 | 1 | 1 | 9 |
| 9 | 1 | −15694499 | 482751380 | 15 | 3 | −1 | 18 |
| 9 | 4 | −2369871 | 802219205 | 15 | 14 | 1 | 18 |
| 9 | 5 | 1196310440 | 2727545297 | 16 | 1 | −9379980173 | 534775120 |
| 9 | 6 | −12236093 | 48275138 | 16 | 2 | 493920 | 393217 |
| 9 | 7 | 112607 | 2839714 | 16 | 3 | 92610 | 393217 |
| 9 | 8 | −1241317 | 96550276 | 16 | 4 | 16192827 | 222167605 |
| 10 | 1 | 53529428429 | 45634027520 | 16 | 5 | −16719820312 | 2266109571 |
| 10 | 4 | 7464384207 | 151666032640 | 16 | 6 | −2287318770977 | 19773310062 |
| 10 | 5 | −466500093865 | 96687095808 | 16 | 7 | −5088054879 | 268173994 |
| 10 | 6 | 251090729 | 100663296 | 16 | 8 | 145283084168551 | 7092936326608 |
| 10 | 7 | −1972264857 | 2952790016 | 16 | 9 | 2267252583356677 | 19312852955000 |
| 10 | 8 | 5553817927 | 25232932864 | 16 | 10 | 1593601199469232128 | 61523723464121875 |
| 10 | 9 | 218123025995 | 138781130752 | 16 | 11 | −59691112677 | 9830425000 |
| 11 | 1 | 95553638 | 44883315 | 16 | 12 | 4309516947456 | 7718977505525 |
| 11 | 4 | 1325104 | 11049705 | 16 | 13 | 5558223321 | 2494568648 |
| 11 | 5 | −3865665440 | 338120973 | 16 | 14 | −92610 | 393217 |
| 11 | 6 | −71919334 | 86774409 | 16 | 15 | −493920 | 393217 |
| 11 | 7 | −97709266 | 60020433 | 17 | 1 | 887 | 14112 |
| 11 | 8 | 27036181984 | 35017962363 | 17 | 2 | 1 | 20 |
| 11 | 9 | 20183684860 | 1910973141 | 17 | 3 | 1 | 40 |
| 11 | 10 | 268435456000 | 275804802441 | 17 | 6 | 120528 | 50575 |
| 12 | 1 | 9506527619197 | 1855976898560 | 17 | 7 | −34208 | 107415 |
| 12 | 4 | −6581911077 | 192762306560 | 17 | 8 | 11166419537 | 24351602400 |
| 12 | 5 | 423361032185 | 122885970432 | 17 | 9 | −70322610762721 | 30323601000000 |
| 12 | 6 | 2238520211347 | 59363966976 | 17 | 10 | −140737488355328 | 3747412280390625 |
| 12 | 7 | 1247928517773 | 232679456768 | 17 | 11 | 528039 | 8750000 |
| 12 | 8 | −471699954635119 | 85178422132736 | 17 | 12 | 43670044672 | 216424333125 |
| 12 | 9 | −276460201705587 | 7055441592320 | 17 | 13 | 4201227 | 8881600 |
| 12 | 10 | −615042605056 | 77689354975 | 17 | 14 | −1 | 40 |
| 12 | 11 | 520936699437 | 272937779200 | 17 | 15 | −1 | 20 |
| 13 | 1 | 3919624801 | 9998920260 | 17 | 16 | 393217 | 11113200 |

Much of the previous development of discrete embedded RKq(p) pairs has involved the simultaneous solution of the RK equations of condition for both orders q and p where usually $p = q - 1$. Except for low orders the necessity of embedding increased the number of stages required from that of $D_q$ given in Table 1. The strategy in this work is to initially ignore the requirement for a lower order embedded process and to develop discrete and continuous RK processes of order $q^*$ with $s^*$ common stages. The RK equations of condition for the continuous process may be written [9]

$$\Phi B = G$$

where $\Phi$ is a matrix with $N_{q^*}$ rows and $s^*$ columns and is dependent on the RK parameters, $c_i$ and $a_{ij}$, $B$ is a matrix with $s^*$ rows and $q^*$ columns whose elements are $B_{ij}$ where $b_i^*(\sigma) = \sum_{j=0}^{q^*-1} B_{ij}\ \sigma^j,\ i = 1, 2, \ldots, s^*$ and $G$ is a matrix with constant coefficients that are associated with the RK equations of condition. From Owren and Zennaro [11] it is required that

$$rank(\Phi) = rank(\Phi \mid G) \tag{5}$$

For each process of order $q^*$ the equations of condition will be solved using $s^*$ stages to obtain a continuous process. The discrete process, of order $q = q^*$, will then be obtained by satisfying the FSAL conditions (2) which will ensure $C^1$ smoothness of the continuous process. The lower order process (or processes if "stretched" estimation is used) is then considered so that local error estimation can be obtained. The free parameters are then chosen with small $A^{(q+1)}$ as the main priority but with consideration also given to the size of the parameters, to the affect on the local error estimation process and to the size of $J^{(q^*+1)}$ where

$$J^{(q^*+1)} = \sqrt{\int_0^1 \sum_{j=1}^{N_{q^*+1}-N_{q^*}} [\sigma^{q^*+1}\tau_j^{*(q^*+1)} - \sigma\widehat{\tau}_j^{(q^*+1)}]^2 \ d\sigma} \tag{6}$$

which is an easier to compute comparable measure to $I_B$. An additional point to note regarding the lower order discrete embedded process is that the error estimation process should preferably involve the function evaluation at the endpoint of the step so that the situation as outlined by Hairer et al. [9] in relation to the RK8(6) process of [5] does not occur.

# 2 RK triple processes

Owren and Zennaro [12] have developed triple processes of orders 3, 4 and 5 in which $I_C$ has been minimised and which have been tested on a selection of problems from the DETEST package [10] using an efficiency criterion based on expected global accuracy. These triples will be considered and new triples of orders $3, 4, \ldots, 9$ will be developed but all testing will be carried out using the efficiency criterion of actual global accuracy against cost.

## 2.1 Order 3

From Table 1 $s^* = 4$ is required and the 4 equations of condition for the continuous process are easily solved [12]. The FSAL assumption (2) then leads to a 2-parameter ($c_2$ and $c_3$) family of discrete/continuous processes. As usual with such models there are certain restrictions on the values of $c_2$ and $c_3$. The expressions for the principal truncation error terms for the continuous process are

$$\tau_1^{*(4)} = \frac{-3\sigma^2 + \sigma(12c_2c_3 - 8(c_2 + c_3) + 12) - 18c_2c_3 + 12(c_2 + c_3) - 12}{72\sigma^2},$$

$$\tau_2^{*(4)} = \frac{-3\sigma^2 - 4\sigma(2c_3 - 3) - 12(1 - c_3)}{24\sigma^2},$$

$$\tau_3^{*(4)} = \frac{-\sigma^2 + 4\sigma(1 - c_2) - 2(2 - 3c_2)}{24\sigma^2} \quad \text{and} \quad \tau_4^{*(4)} = \frac{-(\sigma - 2)^2}{24\sigma^2}.$$

The corresponding expressions for the principal truncation error terms of the discrete process can be obtained by putting $\sigma = 1$ which yields the following expression for $A^{(4)} = \|\hat{\boldsymbol{\tau}}^{(4)}\|_2$

$$A^{(4)} = \sqrt{\frac{(9c_2^2 - 12c_2 + 40)c_3^2 - (12c_2^2 - 17c_2 + 60)c_3 + 13c_2^2 - 15c_2 + 27}{1296}}$$

Choosing $(c_2, c_3) = (\frac{1}{2}, \frac{3}{4})$ gives a near minimal value for $A^{(4)}$ of $\approx 4.18 \times 10^{-2}$ with $J^{(4)} \approx 6.43 \times 10^{-3}$. To complete the triple it is easiest to obtain a discrete formula of order 2 which requires the solution of 2 equations of condition. This involves either 3 or 4 unknown weights ($b_i$) depending on whether or not the FSAL evaluation is utilised. Owren and Zennaro [12] suggest that $b_4$ should be zero so that the FSAL evaluation is not used for the error estimation process and therefore one less evaluation will be lost in the case of any step rejections. However, if neither $c_2$ or $c_3$ is 1, then without the FSAL evaluation the error estimation process could lead to poor results on problems such as EULR [9] and, in any case, it may possibly be preferable to utilise the FSAL evaluation if better error estimation is obtained. A careful choice of the free weights (with $b_4 \neq 0$) gives the triple process in Table 3. Owren and Zennaro [12] in looking for a small value for $I_C$ chose $(c_2, c_3) = (\frac{12}{23}, \frac{4}{5})$

and $b_4 = 0$ for their triple giving $A^{(4)} \approx 4.26 \times 10^{-2}$ and $J^{(4)} \approx 6.46 \times 10^{-3}$. It might be expected therefore that, from the global error/cost efficiency point of view, there would be little difference in general between the two triples.

Table 3: RKT3(2)3

| $c_i$ | $a_{ij}$ | | | $\widehat{b}_i$ | $b_i$ | $b_i^*$ |
|---|---|---|---|---|---|---|
| $0$ | | | | $\frac{2}{9}$ | $\frac{7}{36}$ | $\frac{5\sigma^2-12\sigma+9}{9}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | | | $\frac{1}{3}$ | $\frac{19}{36}$ | $\frac{\sigma(3-2\sigma)}{3}$ |
| $\frac{3}{4}$ | $0$ | $\frac{3}{4}$ | | $\frac{4}{9}$ | $\frac{1}{6}$ | $\frac{4\sigma(3-2\sigma)}{9}$ |
| $1$ | $\frac{2}{9}$ | $\frac{1}{3}$ | $\frac{4}{9}$ | $0$ | $\frac{1}{9}$ | $\sigma(\sigma-1)$ |

## 2.2 Order 4

From Table 1 $s^* = 6$ is required and so the 8 equations of condition for the continuous process can now be solved. The FSAL assumption (2) leaves the following free parameters: $c_2$, $c_3$, $c_4$, $c_5$ and $a_{32}$. Unfortunately the expressions for some of the $a_{ij}$ are exceedingly large. It is however possible to make progress by choosing $Q_{31} = 0$ where

$$Q_{ik} = \sum_{j=1}^{i-1} a_{ij} c_j^k - \frac{c_i^{k+1}}{k+1}, \quad i = 2, 3, \ldots, s^*, \quad k = 1, 2, \ldots \tag{7}$$

and this has the affect of making $Q_{i1} = 0,\ i = 4, 5$ and $\widehat{b}_2 = b_2^* = 0$ which greatly simplifies the analysis. Examination of the principal truncation error terms for the discrete process now shows that only $\widehat{\tau}_4^{(5)}$ and $\widehat{\tau}_8^{(5)}$ are dependent on $c_2$ which can only therefore play a minor role in the determination of $A^{(5)}$. Hence the standard choice of $c_2 = \frac{2}{3} c_3$ is made giving

$$\widehat{\tau}_1^{(5)} = \frac{30c_3c_4c_5 - 20(c_3c_4 + c_4c_5 + c_5c_3) + 15(c_3 + c_4 + c_5) - 12}{1440},$$

$$\widehat{\tau}_2^{(5)} = 6\widehat{\tau}_1^{(5)}, \quad \widehat{\tau}_3^{(5)} = 3\widehat{\tau}_1^{(5)}, \quad \widehat{\tau}_4^{(5)} = 4\widehat{\tau}_1^{(5)}, \quad \widehat{\tau}_7^{(5)} = 3\widehat{\tau}_5^{(5)}, \quad \widehat{\tau}_8^{(5)} = \widehat{\tau}_5^{(5)},$$

$$\widehat{\tau}_5^{(5)} = -\frac{10(c_3c_4 - 5(c_3 + c_4) + 3)}{360}, \quad \widehat{\tau}_9^{(5)} = -\frac{(10c_4c_5 - 5c_4 + 1)}{120}$$

$$\text{and}\quad \widehat{\tau}_6^{(5)} = \frac{30c_3c_4c_5 - 20c_4(c_3+c_5) + 5c_5 + 15c_4 - 4}{120}.$$

The expressions for the $\tau_j^{*(5)},\ j = 1, 2, \ldots, 9$ are similarly related but involve $\sigma$ and are much larger and therefore not given here. It turns out that $A^{(5)} = 0$ with $c_3 = \frac{2}{5}$, $c_4 = 1$ and $c_5 = 1$. Unfortunately $c_4 \neq c_5$ is one of the restrictions for this 3-parameter family. With small $A^{(5)}$ as a priority a reasonable choice is $c_3 = \frac{2}{5}$, $c_4 = \frac{24}{25}$ and $c_5 = 1$ giving $A^{(5)} \approx 6.37 \times 10^{-4}$ and $J^{(5)} \approx 3.85 \times 10^{-3}$. For error estimation a discrete process of order 3 can be obtained by solving 4 equations involving the 6 weights $b_i,\ i = 1, 2, \ldots, 6$. Practical tests show that there is no obvious advantage in utilising the FSAL evaluation in the error estimation process and a suitable triple is given in Table 4. Owren and Zennaro [12] chose $c_2 = \frac{1}{6}$, $c_3 = \frac{11}{37}$, $c_4 = \frac{11}{17}$, $c_5 = \frac{13}{15}$ and $b_6 = 0$ giving $A^{(5)} \approx 3.16 \times 10^{-3}$ with $J^{(5)} \approx 1.74 \times 10^{-3}$ and so their error estimation process does not involve the endpoint evaluation. The difference in the $A^{(5)}$ values is such that it might be expected, from a global error/cost point of view, that there would be a significant difference in the practical results particularly at the more stringent tolerances.

Table 4: RKT4(3)4

| $c_i$ | $a_{ij}$ | | | | | $\widehat{b}_i$ | $b_i$ | $b_i^*$ |
|---|---|---|---|---|---|---|---|---|
| $0$ | | | | | | $\frac{73}{576}$ | $\frac{1}{12}$ | $-\frac{375\sigma^3-1180\sigma^2+1308\sigma-576}{576}$ |
| $\frac{4}{15}$ | $\frac{4}{15}$ | | | | | $0$ | $0$ | $0$ |
| $\frac{2}{5}$ | $\frac{1}{10}$ | $\frac{3}{10}$ | | | | $\frac{575}{1008}$ | $\frac{25}{36}$ | $\frac{25\sigma(75\sigma^2-196\sigma+144)}{1008}$ |
| $\frac{24}{25}$ | $\frac{528}{625}$ | $-\frac{1944}{625}$ | $\frac{2016}{625}$ | | | $\frac{3125}{4032}$ | $0$ | $-\frac{3125\sigma(3\sigma-2)(5\sigma-6)}{4032}$ |
| $1$ | $\frac{337}{272}$ | $-\frac{645}{136}$ | $\frac{2175}{476}$ | $-\frac{125}{1904}$ | | $-\frac{17}{36}$ | $\frac{2}{9}$ | $\frac{17\sigma(705\sigma^2-1324\sigma+576)}{1548}$ |
| $1$ | $\frac{73}{576}$ | $0$ | $\frac{575}{1008}$ | $\frac{3125}{4032}$ | $-\frac{17}{36}$ | $0$ | $0$ | $\frac{\sigma(\sigma-1)(115\sigma-72)}{43}$ |

## 2.3 Order 5

From Table 1 $s^* = 8$ is required. The solution of the 17 equations associated with the continuous process has been investigated in detail by Owren and Zennaro ([11] and [12]). Together with the FSAL conditions (2) this leads of a family of processes with $c_2$, $c_3$, $c_6$, $c_7$ and $a_{54}$ as the free parameters. It should be noted that the model requires

$$c_4 = c_5 = 2c_3, \quad b_2^* = \widehat{b}_2 = 0, \quad Q_{i1} = 0,\ i = 1, 3, 4, \ldots, s^* \tag{8}$$

Consideration of the expressions for the 20 principal truncation error terms in the fifth order discrete process shows that none are dependent on $a_{54}$, only $\widehat{\tau}_j^{(6)}$, $j = 4, 5, 13, 14, 19$ are dependent on $c_2$ and that $\widehat{\tau}_j^{(6)}$, $j = 15, 18$ are only dependent on $c_3$. Investigations show that the smallest possible value of $A^{(6)}$ is approximately $9.2 \times 10^{-4}$. Choosing $c_2 = \frac{1}{5}$, $c_3 = \frac{3}{10}$, $c_6 = \frac{2}{5}$, $c_7 = \frac{9}{10}$ gives $A^{(6)} \approx 9.53 \times 10^{-4}$ with $J^{(6)} \approx 9.04 \times 10^{-4}$ and a suitable fourth order discrete embedded formula is easily found leading to the triple in Table 5. The 5(4) formula of Owren and Zennaro [12] has $c_2 = \frac{1}{6}$, $c_3 = \frac{1}{4}$, $c_6 = \frac{9}{14}$, $c_7 = \frac{7}{8}$ and $b_8 = 0$ giving $A^{(6)} \approx 1.09 \times 10^{-3}$ with $J^{(6)} \approx 1.92 \times 10^{-4}$ and so it might be expected therefore that, from the global error/cost efficiency point of view, there would be little difference in general between the two triples.

It is interesting to compare these triples, as far as the cost is concerned, with the RK5(4)7FM (sometimes referred to as DOPRI5) of [3] which uses 7 stages (including the FSAL stage) per step with $A^{(6)} \approx 3.99 \times 10^{-4}$ and which was originally developed without any continuous process. Subsequently it was provided with both a fourth order continuous process using the 7 stages [4] and a fifth order continuous process using 9 stages [1].

Table 5: RKT5(4)5

| $c_i$ | $a_{ij}$ | | | | | | | $\widehat{b}_i$ | $b_i$ | $b_i^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $0$ | | | | | | | | $\frac{29}{324}$ | $\frac{73}{1620}$ | $\frac{(1895\sigma^4-5080\sigma^3+5007\sigma^2-2412\sigma+648)}{648}$ |
| $\frac{1}{5}$ | $\frac{1}{5}$ | | | | | | | $0$ | $0$ | $0$ |
| $\frac{3}{10}$ | $\frac{3}{40}$ | $\frac{9}{40}$ | | | | | | $\frac{5}{9}$ | $\frac{11}{9}$ | $-\frac{5\sigma(185\sigma^3-415\sigma^2+281\sigma-54)}{27}$ |
| $\frac{3}{5}$ | $\frac{3}{10}$ | $-\frac{9}{10}$ | $\frac{6}{5}$ | | | | | $\frac{25}{162}$ | $-\frac{185}{3078}$ | $\frac{25\sigma(35\sigma^3-55\sigma^2+3\sigma+18)}{162}$ |
| $\frac{3}{5}$ | $\frac{1}{10}$ | $0$ | $\frac{2}{5}$ | $\frac{1}{10}$ | | | | $\frac{35}{162}$ | $\frac{1432}{1539}$ | $-\frac{5\sigma(2785\sigma^3-4640\sigma^2+981\sigma+846)}{648}$ |
| $\frac{2}{5}$ | $\frac{8}{135}$ | $\frac{4}{15}$ | $\frac{8}{135}$ | $\frac{2}{45}$ | $-\frac{4}{135}$ | | | $-\frac{1}{4}$ | $-\frac{5}{4}$ | $\frac{\sigma(365\sigma^3-760\sigma^2+429\sigma-36)}{8}$ |
| $\frac{9}{10}$ | $-\frac{14}{95}$ | $\frac{261}{760}$ | $\frac{196}{95}$ | $-\frac{17}{380}$ | $\frac{27}{20}$ | $-\frac{405}{152}$ | | $\frac{19}{81}$ | $\frac{1}{81}$ | $\frac{19\sigma(5\sigma^3+5\sigma^2-27\sigma+18)}{81}$ |
| $1$ | $\frac{29}{324}$ | $0$ | $\frac{5}{9}$ | $\frac{25}{162}$ | $\frac{35}{162}$ | $-\frac{1}{4}$ | $\frac{19}{81}$ | $0$ | $\frac{1}{10}$ | $\frac{\sigma(\sigma-1)(5\sigma^2-15\sigma+18)}{8}$ |

### 2.4 Order 6

From Table 1 $s^* = 11$ is required and the solution of the 37 equations of condition for the continuous process utilises the requirements (8) and leads to a significant restriction in that either

$$a_{76} = 0 \tag{9}$$

or

$$c_7 = c_6 \tag{10}$$

At this point in the analysis the continuous process is of order 6 with degrees of freedom $c_2$, $c_5$, $c_6$, $c_8$, $c_9$, $c_{10}$, $c_{11}$, $a_{54}$, $a_{82}$, $a_{92}$, $a_{98}$, $a_{102}$, $a_{108}$, $a_{109}$, $a_{112}$, $a_{118}$, $a_{119}$, $a_{1110}$ and either $c_7$ or $a_{76}$. It should be noted that (9) and (10) cannot both be true. In either case after applying the FSAL assumption (2) and consideration of the discrete process there are still a considerable number of degrees of freedom left. Since small $A^{(7)}$ is the priority these are chosen accordingly. In fact it is found that $A^{(7)}$ can be made zero. Thus $\widehat{\tau}_j^{(7)} = 0, j = 1, 2, \ldots, 85$, making the discrete process of order 7. It should be noted that $\widehat{\tau}_{15}^{(7)}$ is rather difficult to deal with since, after applying (8), it involves a term of the form $\sum_{i=2}^{10} \widehat{b}_i a_{i2}^2$. Although a model is possible based on either (9) or (10) the analysis is far easier if (10) is true and for this model the following are satisfied

$$\begin{aligned} a_{54} = \frac{c_5}{6}, \quad \widehat{b}_i = b_i^* = b_i = 0,\ i = 2, 3, 4, 6, \\ Q_{i1} = Q_{i2} = 0,\ i = 3, 4, \ldots, s^*, \\ Q_{i3} = a_{i2} = 0,\ i = 5, 7, 8, \ldots, s^*, \end{aligned} \tag{11}$$

and

$$\widehat{R}_{0j} = 0,\ j = 1, 2, \ldots, s, \tag{12}$$

where

$$\widehat{R}_{kj} = \sum_{i=j+1}^{s} \widehat{b}_i c_i^k a_{ij} - \frac{\widehat{b}_j (1 - c_j^{(k+1)})}{k+1}, \quad j = 1, 2, \ldots, s, \quad k = 0, 1, \ldots \tag{13}$$

and where $c_9$ is restricted as follows

$$c_9 = \frac{35c_5c_6c_8 - 21(c_5c_6 + c_6c_8 + c_8c_5) + 14(c_5 + c_6 + c_8) - 10}{7(10c_5c_6c_8 - 5(c_5c_6 + c_6c_8 + c_8c_5) + 3(c_5 + c_6 + c_8) - 2)}$$

Note that $\widehat{R}_{010} = 0$ implies $c_{10} = 1$ leaving the remaining degrees of freedom as $c_5$, $c_6$ and $c_8$. Now since the higher order discrete process has been made to be order 7, implying that $I_C$ and $I_B$ are identical, the question arises as to how these remaining degrees of freedom should be chosen. It seems sensible to look for a small value of $J^{(7)}$ with possibly a small value of $A^{(8)}$. The choice $c_5 = \frac{1}{2}$, $c_6 = \frac{3}{16}$ and $c_8 = \frac{3}{5}$ implies $c_9 = \frac{6}{7}$ and gives $J^{(7)} \approx 7.18 \times 10^{-5}$ and $A^{(8)} \approx 5.68 \times 10^{-5}$. With regard to the error estimation process it is not possible to obtain a distinct embedded sixth order formula using the 11 stages. This is not a significant problem since the embedded formula is only used to monitor and predict step sizes. An embedded fifth order formula is easily obtained or alternatively a combination of lower order formulae is possible so that stretched estimation can be used. A triple utilising a fifth order embedded formula is given in Table 6 which contains only the non-zero entities.

It is interesting to compare this triple, as far as the cost is concerned, with the RK6(5)9FM of [6] which yields sixth order discrete aproximations and error estimates in 9 evaluations per step (including the FSAL evaluation). A tenth evaluation per step is required when using the fifth order continuous process and a total of 12 evaluations per step are necessary if sixth order continuous approximations are required [1]. The RKT7(5)6 which requires 10 evaluations per step for seventh order discrete approximations and error estimates and requires the additional FSAL evaluation per step for sixth order continuous approximations would appear to be more cost effective.

Table 6: RKT7(5)6. Note that $c_i = \sum_{j=1}^{i-1} a_{ij}$, $b_i^*(\sigma) = \sum_{k=0}^{5} B_{ik}\sigma^k,\ i = 1, 2, \ldots, 11$ and $\widehat{b}_j = a_{11j},\ j = 1, 2, \ldots, 10$.

| $i$ | $j$ | $Numerator(a_{ij})$ | $Denominator(a_{ij})$ | $i$ | $j$ | $Numerator(a_{ij})$ | $Denominator(a_{ij})$ |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 6 | 8 | 6 | −14592 | 78125 |
| 3 | 1 | 1 | 16 | 8 | 7 | 33792 | 78125 |
| 3 | 2 | 3 | 16 | 9 | 1 | 51237 | 285719 |
| 4 | 1 | 1 | 4 | 9 | 3 | 4580136 | 2000033 |
| 4 | 2 | −3 | 4 | 9 | 4 | −419616 | 2000033 |
| 4 | 3 | 1 | 1 | 9 | 5 | −2586870 | 2000033 |
| 5 | 1 | 1 | 12 | 9 | 6 | −8901120 | 2000033 |
| 5 | 3 | 1 | 3 | 9 | 7 | 3840000 | 1294139 |
| 5 | 4 | 1 | 12 | 9 | 8 | 30234375 | 22000363 |
| 6 | 1 | 9 | 128 | 10 | 1 | −1396 | 1647 |
| 6 | 2 | 135 | 1024 | 10 | 3 | −19396 | 1281 |
| 6 | 3 | −3 | 256 | 10 | 4 | 944 | 1281 |
| 6 | 4 | −57 | 4096 | 10 | 5 | 105447 | 10675 |
| 6 | 5 | 45 | 4096 | 10 | 6 | 494336 | 19215 |
| 7 | 1 | 129 | 2048 | 10 | 7 | −213085184 | 15852375 |
| 7 | 3 | −117 | 1024 | 10 | 8 | −2575625 | 380457 |
| 7 | 4 | 207 | 16384 | 10 | 9 | 530621 | 617625 |
| 7 | 5 | −135 | 16384 | 11 | 1 | 179 | 3240 |
| 7 | 6 | 15 | 64 | 11 | 5 | 88 | 375 |
| 8 | 1 | 36 | 625 | 11 | 7 | 2097152 | 7239375 |
| 8 | 3 | 36 | 625 | 11 | 8 | 3125 | 21384 |
| 8 | 4 | −288 | 15625 | 11 | 9 | 285719 | 1215000 |
| 8 | 5 | 4023 | 15625 | 11 | 10 | 61 | 1560 |

| $i$ | $k$ | $Numerator(B_{ik})$ | $Denominator(B_{ik})$ | $i$ | $k$ | $Numerator(B_{ik})$ | $Denominator(B_{ik})$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 8 | 3 | 1559375 | 7128 |
| 1 | 1 | −67 | 12 | 8 | 4 | −59375 | 297 |
| 1 | 2 | 392 | 27 | 8 | 5 | 175000 | 2673 |
| 1 | 3 | −4205 | 216 | 9 | 1 | −16807 | 4500 |
| 1 | 4 | 587 | 45 | 9 | 2 | 16807 | 675 |
| 1 | 5 | −280 | 81 | 9 | 3 | −4823609 | 81000 |
| 5 | 1 | −432 | 25 | 9 | 4 | 1025227 | 16875 |
| 5 | 2 | 528 | 5 | 9 | 5 | −134456 | 6075 |
| 5 | 3 | −5448 | 25 | 10 | 1 | 19764 | 302107 |
| 5 | 4 | 23696 | 125 | 10 | 2 | −160857 | 604214 |
| 5 | 5 | −896 | 15 | 10 | 3 | −2003423 | 2416856 |
| 7 | 1 | 524288 | 53625 | 10 | 4 | 3669821 | 1510535 |
| 7 | 2 | −3670016 | 96525 | 10 | 5 | −1233176 | 906321 |
| 7 | 3 | 29360128 | 482625 | 11 | 1 | 45225 | 46478 |
| 7 | 4 | −12058624 | 268125 | 11 | 2 | −157380 | 23239 |
| 7 | 5 | 3670016 | 289575 | 11 | 3 | 844295 | 46478 |
| 8 | 1 | 3125 | 198 | 11 | 4 | −485844 | 23239 |
| 8 | 2 | −59375 | 594 | 11 | 5 | 198464 | 23239 |

| $i$ | 1 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $Numerator(b_i)$ | 575 | 1 | 1 | −1 | −128 | 45568 | 34625 | 31213 | 175 |
| $Denominator(b_i)$ | 6912 | 2 | 40 | 5 | 225 | 96525 | 76032 | 172800 | 3328 |

## 2.5 Order 7

From Table 1 there are 85 equations of condition associated with a continuous process of order 7 and this section briefly outlines a solution of the equations with $s^* = 14$. The solution builds upon the analysis from the previous orders and assumes the FSAL conditions (2) together with the equations (8), (10) and (11). The first requirement of the solution is that either

$$c_9 = c_8 \quad \text{and} \quad a_{98} \neq 0 \quad \text{or} \quad a_{98} = 0 \quad \text{and} \quad c_9 \neq c_8$$

or

$$c_8 = c_3 \tag{14}$$

The analysis is greatly simplified by assuming (14) and this then leads to the following requirement that either (but not both)

$$c_{10} = c_9 \tag{15}$$

or

$$a_{109} = 0 \tag{16}$$

Following the model of order 6 it is possible using equation (12) to make the discrete formula of order 8. Having done that leaves the remaining degrees of freedom as $c_3$, $c_6$, $c_9$, $c_{11}$, $c_{12}$, $a_{1413}$ $(\widehat{b}_{13})$ and either $a_{109}$ or $c_{10}$ depending on whether equation (15) or equation (16) is imposed. It is not obvious as to how $a_{1413}$ should best be selected so it has been chosen to zeroise $\widehat{\tau}_1^{(9)}$ and it would seem to be more flexible to impose equation (16) and leave $c_{10}$ free. This then permits the remaining free parameters to be chosen to give small values for $J^{(8)}$ and $A^{(9)}$. Choosing $c_3 = \frac{7}{50}$, $c_6 = \frac{19}{40}$, $c_9 = \frac{2}{25}$, $c_{10} = \frac{8}{15}$, $c_{11} = \frac{4}{5}$ and $c_{12} = \frac{22}{25}$ gives $J^{(8)} \approx 6.28 \times 10^{-6}$ and $A^{(9)} \approx 4.48 \times 10^{-6}$. Similar to the situation of the previous section it is not possible to obtain a distinct embedded seventh order formula using the 14 common stages but a sixth order formula is easily found or alternatively a combination of lower order formulae can be obtained so that stretched estimation may be employed. A triple utilising a sixth order embedded process can be found in Tables 7 and 8 which contain only the non-zero entities.

It is instructive to compare the cost of this triple with the RK8(7)13M of [13] and the RK8(6)12M of [5] (equivalently the DOP853 of [9]). The RK8(7)13M and RK8(6)12M use 13 and 12 stages respectively for discrete approximations and error estimation and then both use either an additional 4 or 7 stages (which includes the FSAL stage) for continuous approximation depending on whether $q^* = 7$ or $q^* = 8$ [1]. The RKT8(6)7 uses 13 stages for discrete estimation with the additional FSAL stage for seventh order continuous approximation and thus appears to be more cost effective.

Table 7: RKT8(6)7 which is continued in Table 8. Note that $\widehat{b}_i = a_{14i}$, $i = 1, 2, \ldots, 13$ and $b_i = a_{15i}$, $c_i = \sum_{j=1}^{i-1} a_{ij}$, $i = 1, 2, \ldots, 14$.

| $i$ | $j$ | $a_{ij}$ | $i$ | $j$ | $a_{ij}$ |
|---|---|---|---|---|---|
| 2 | 1 | 9.3333333333333333333333333333333E-2 | 11 | 7 | -7.1579553906147432099663790337E0 |
| 3 | 1 | 3.5E-2 | 11 | 8 | 2.2659358154705277003480060595E0 |
| 3 | 2 | 1.05E-1 | 11 | 9 | 1.31756939077540917088647769884E1 |
| 4 | 1 | 1.4E-1 | 11 | 10 | 4.35147985890989745154013147945E0 |
| 4 | 2 | -4.2E-1 | 12 | 1 | 2.13650273135026061952015557422E0 |
| 4 | 3 | 5.6E-1 | 12 | 3 | 1.22726517463837019805397194412E1 |
| 5 | 1 | 4.6666666666666666666666666666667E-2 | 12 | 4 | 4.57653009750266308294439689668E0 |
| 5 | 3 | 1.8666666666666666666666666666667E-1 | 12 | 5 | -1.02693748349496721494231277368E1 |
| 5 | 4 | 4.6666666666666666666666666666667E-2 | 12 | 6 | -3.7629577005440390399992522631E-1 |
| 6 | 1 | 5.51743197278911564625850340136E-1 | 12 | 7 | 5.13017034156407434494186004831E0 |
| 6 | 2 | -1.68355389030612244897959183674E0 | 12 | 8 | 9.62489338117485142433512784366E-2 |
| 6 | 3 | 1.28544855442176870748299319728E0 | 12 | 9 | -1.03442766767566431388674762321E1 |
| 6 | 4 | -1.02147251199586977648202137998E0 | 12 | 10 | -2.59772774629732852847083885323E0 |
| 6 | 5 | 1.3428346506013119533527696793E0 | 12 | 11 | 2.55571177445599178571884809579E-1 |
| 7 | 1 | 6.24295812074829931972789115646E-2 | 13 | 1 | -1.35769285555186849895417607979E0 |
| 7 | 3 | 1.1561591754239009036450401056E-1 | 13 | 3 | -5.1848880697594401985846769119E0 |
| 7 | 4 | -2.48173453254069484936831875607E-1 | 13 | 4 | -9.44146275842569180247520769719E-1 |
| 7 | 5 | 4.76004820175838192419825072886E-1 | 13 | 5 | 5.7895230679047775815740126675E0 |
| 7 | 6 | 6.9123134328358208955223880597E-2 | 13 | 6 | -8.78331322343203825844350417575E-2 |
| 8 | 1 | 5.31754385964912280701754385965E-2 | 13 | 7 | -5.3269980917879320756691248069E0 |
| 8 | 3 | 1.07960199004975124378109452736E-1 | 13 | 8 | -3.29419214246872762413611365341E0 |
| 8 | 4 | -4.55448717948717948717948717949E-2 | 13 | 9 | 7.12498180803033672915227750425E0 |
| 8 | 5 | 2.13141025641025641025641025641E-2 | 13 | 10 | 4.23294185993291744523268528268E0 |
| 8 | 6 | 1.80954510721537509608660353975E-2 | 13 | 11 | -4.04195634685480312241204160092E-1 |
| 8 | 7 | -1.50003194428508726399201574995E-2 | 13 | 12 | 4.52499466462306516458275969143E-1 |
| 9 | 1 | 4.66308628714643752237737200143E-2 | 14 | 1 | 1.01794429109647960346134352579E-2 |
| 9 | 3 | 5.89926174413351732228314575591E-2 | 14 | 5 | 3.42368385444252768240392979897E-1 |
| 9 | 4 | -3.85681391925321394464275175145E-2 | 14 | 7 | -2.93579170210195254952533184107E-1 |
| 9 | 5 | 2.31207294594081206081229649243E-2 | 14 | 8 | -9.95353542504925310796528261447E-2 |
| 9 | 6 | 1.33009364423326992630144289062E-2 | 14 | 9 | 2.07437934572206346408597479637E-1 |
| 9 | 7 | -1.12383747568330718937876881162E-2 | 14 | 10 | 5.43021201240774613165496369804E-1 |
| 9 | 8 | -1.22386322651751569775273657731E-2 | 14 | 11 | 1.07235531414217311653209089107E-1 |
| 10 | 1 | 5.44217687074829931972789115646E-2 | 14 | 12 | 1.44540738194287577003626386342E-1 |
| 10 | 3 | 5.40333156097836021965644146011E-1 | 14 | 13 | 3.8331290683984373526250270207E-2 |
| 10 | 4 | 1.13688418211922452259942034977E-1 | 15 | 1 | -5.91293258004837089685158123242E-2 |
| 10 | 5 | 7.13352588752466984012721183007E-2 | 15 | 5 | -1.14901053624678659268733720599E0 |
| 10 | 6 | 5.33742685221169022273976869374E-3 | 15 | 7 | 3.77974861322618360761792589719E0 |
| 10 | 7 | 1.41741045931070379958169549861E-1 | 15 | 8 | 7.48570483449616205367366755746E-1 |
| 10 | 8 | -3.93523741342436902671713196074E-1 | 15 | 9 | 1.74577267785426015996335159381E-2 |
| 11 | 1 | -2.5940248582906658528380592843E0 | 15 | 10 | -3.00845790812659142174173541204E0 |
| 11 | 3 | -1.77182396161676339315977974886E1 | 15 | 11 | 9.70820946719519308812662261486E-1 |
| 11 | 4 | -7.20291569558395826074056304001E0 | 15 | 12 | -4.0E-1 |
| 11 | 5 | 1.49881506312855377166637034571E1 | 15 | 13 | 1.0E-1 |
| 11 | 6 | 6.91875347236946677726180862128E-1 | | | |

Table 8: Continuation of the RKT8(6)7 from Table 7. Note that $b_i^*(\sigma) = \sum_{k=0}^{6} B_{ik}\sigma^k,\ i = 1, 2, \ldots, 14.$

| $i$ | $k$ | $B_{ik}$ | $i$ | $k$ | $B_{ik}$ |
|---|---|---|---|---|---|
| 1 | 0 | 1.0E0 | 10 | 1 | 9.65701535222266902447206365602E-1 |
| 1 | 1 | -1.41239093743887554093438640359E1 | 10 | 2 | 1.29274299904241623819239369537E1 |
| 1 | 2 | 8.4016766195696458031461586846E1 | 10 | 3 | -1.00480323438077279181257364969E2 |
| 1 | 3 | -2.51083997467010984848468599308E2 | 10 | 4 | 2.45834235483979601485491021852E2 |
| 1 | 4 | 3.86512292414528095658571249453E2 | 10 | 5 | -2.42964579882849927174983253853E2 |
| 1 | 5 | -2.91148854238488538278621434142E2 | 10 | 6 | 8.42605575125419501995439500197E1 |
| 1 | 6 | 8.48378819125746896424356746212E1 | 11 | 1 | 8.35557991425411575752552757343E-2 |
| 5 | 1 | 5.10665844292158893795819184009E0 | 11 | 2 | -3.81989088470777655385269085515E-1 |
| 5 | 2 | -4.00124279090280806878089311459E1 | 11 | 3 | 1.70967961712293483156233961215E0 |
| 5 | 3 | 1.84440467426588995836723836551E2 | 11 | 4 | -5.28883927213581506454906269727E0 |
| 5 | 4 | -3.78973592698221239000827838379E2 | 11 | 5 | 7.30946577097275164964837014524E0 |
| 5 | 5 | 3.41538781236289637930611683348E2 | 11 | 6 | -3.32463729521741760719842416123E0 |
| 5 | 6 | -1.11757518113106650248416549234E2 | 12 | 1 | -6.33380888284421342020961449909E-2 |
| 7 | 1 | -2.0751485626953971018794503634E0 | 12 | 2 | -1.09526815610176510133599623253E0 |
| 7 | 2 | -9.92180312887292232371883540967E0 | 12 | 3 | 8.61906649612399098614496017804E0 |
| 7 | 3 | 9.25598164917894535323732388042E1 | 12 | 4 | -2.30612339267248796358151679261E1 |
| 7 | 4 | -2.15979497620388810544292166846E2 | 12 | 5 | 2.59748166009870704285753056776E1 |
| 7 | 5 | 2.02287446902906568511069478446E2 | 12 | 6 | -1.02295021872616869663633791656E1 |
| 7 | 6 | -6.71643932529490873285047978155E1 | 13 | 1 | -2.84608939125612693786991431373E-1 |
| 8 | 1 | -2.14183863541985721594046947605E1 | 13 | 2 | 4.29888914910786433803628796472E-3 |
| 8 | 2 | 2.2543825665248598292363025894E2 | 13 | 3 | 5.20337256488446600050553460014E0 |
| 8 | 3 | -8.84829314979143198157611115581E2 | 13 | 4 | -1.46885666621411381932249338734E1 |
| 8 | 4 | 1.57460856803292263565777875922E3 | 13 | 5 | 1.54411838034484010111998278429E1 |
| 8 | 5 | -1.29008703344712019545777930343E3 | 13 | 6 | -5.63734836553123961550522315598E0 |
| 8 | 6 | 3.96188374740802854662306442791E2 | 14 | 1 | 2.91422009445041363278596489354E-1 |
| 9 | 1 | 3.15180535325053411373578467654E1 | 14 | 2 | 4.38587473682482536753098062593E-1 |
| 9 | 2 | -2.71413850918964647969857885217E2 | 14 | 3 | -8.48742089186228431556770654766E0 |
| 9 | 3 | 9.52348654179583905315594876659E2 | 14 | 4 | 2.37789877586610534396939312211E1 |
| 9 | 4 | -1.59274235351047950380282579202E3 | 14 | 5 | -2.63071727836903908960901174964E1 |
| 9 | 5 | 1.25795594603754462227636944346E3 | 14 | 6 | 1.0285596433764097871932198271E1 |
| 9 | 6 | -3.7745901138561751061022989217E2 | | | |

## 2.6 Order 8

From Table 1 there are 200 equations associated with a process of order 8 and the value of $C_8$ is not precisely known. This section briefly outlines a solution of the equations with $s^* = 18$. The solution builds upon the analysis from the previous orders and assumes the FSAL conditions (2) together with the equations (8), (10), (11), (14) and (16). A significant requirement for the continuous process to be of order 8 is either

$$a_{1110} = 0 \quad \text{or} \quad c_{12} = c_{11} \quad \text{and} \quad a_{1211} \neq 0$$

or

$$a_{1211} = 0 \quad \text{and} \quad c_{12} \neq c_{11} \tag{17}$$

The easiest way forward is to impose equation (17) and at this stage the continuous process is of order 8. There are still a large number of degrees of freedom available and it is possible using equation (12) to make the discrete formula of order 9. In fact considering the discrete formula in isolation it is possible, Hairer [8], to make it order 10 by zeroising certain of the $\widehat{R}_{1j}$ from equation (13). Unfortunately this violates the requirement from equation (5) meaning that the continuous process cannot be of order 8. So having made the discrete process of order 9 the remaining free parameters are chosen to give small values for $J^{(9)}(\approx 1.97 \times 10^{-6})$ and $A^{(10)}(\approx 3.59 \times 10^{-9})$. The resulting triple utilising a seventh order embedded process can be found in Tables 9 and 10 which contain only the non-zero entities.

Table 9: RKT9(7)8 which is continued in Table 10. Note that $\widehat{b}_i = a_{18i}$, $i = 1, 2, \ldots, 17$ and $c_i = \sum_{j=1}^{i-1} a_{ij}$, $i = 1, 2, \ldots, 18$.

| $i$ | $j$ | $a_{ij}$ | $i$ | $j$ | $a_{ij}$ |
|---|---|---|---|---|---|
| 2 | 1 | 1.03703703703703703703703703704E-1 | 13 | 11 | 1.11825158329569318169361499236E-4 |
| 3 | 1 | 3.88888888888888888888888888889E-2 | 13 | 12 | 5.9232826064386410109019245474E-2 |
| 3 | 2 | 1.16666666666666666666666666667E-1 | 14 | 1 | -1.08294884409577446574523573542E1 |
| 4 | 1 | 1.55555555555555555555555555556E-1 | 14 | 3 | -4.58555384056239216218818778683E1 |
| 4 | 2 | -4.66666666666666666666666666667E-1 | 14 | 4 | -1.32193447471870660838992789085E1 |
| 4 | 3 | 6.22222222222222222222222222222E-1 | 14 | 5 | 3.05914618343960445376558308999E1 |
| 5 | 1 | 5.18518518518518518518518518519E-2 | 14 | 6 | 1.90922815696523943781555810831E-1 |
| 5 | 3 | 2.07407407407407407407407407407E-1 | 14 | 7 | -1.88814404888067339077866227303E0 |
| 5 | 4 | 5.18518518518518518518518518519E-2 | 14 | 8 | 2.120320047452681450999642799E1 |
| 6 | 1 | 1.43904006046863189720332577476E0 | 14 | 9 | 1.58374092516843429808128896804E0 |
| 6 | 2 | -4.29081632653061224489795918367E0 | 14 | 10 | 1.01434070966925954939865647766E1 |
| 6 | 3 | 2.79697656840513983371126228269E0 | 14 | 11 | -3.75623558814887119657201098616E-1 |
| 6 | 4 | -3.79589677140697548860814166937E0 | 14 | 12 | -8.5871515129714592091127087542E0 |
| 6 | 5 | 4.49514091350826044703595724004E0 | 14 | 13 | 1.75314464568442281881693067004E1 |
| 7 | 1 | 1.15627362055933484504913076342E-1 | 15 | 1 | 1.23847409176679080375217053279E0 |
| 7 | 3 | -4.18968597540026111454682883254E-2 | 15 | 3 | 1.09333898278372589827844780651E0 |
| 7 | 4 | -1.02048779829392074290033473707E0 | 15 | 4 | 3.31536159352838127791365049203E-1 |
| 7 | 5 | 1.48135325558794946550048590865E0 | 15 | 5 | -2.09936149938613061202956506086E0 |
| 7 | 6 | 1.09848484848484848484848484849E-1 | 15 | 6 | -6.32995160818675610148594925667E-3 |
| 8 | 1 | 6.01213282247765006385696040869E-2 | 15 | 7 | 8.53402652892084128305708186158E-2 |
| 8 | 3 | 1.16077441077441077441077441078E-1 | 15 | 8 | -7.13566595489746263585587501118E-2 |
| 8 | 4 | -4.08333333333333333333333333333E-2 | 15 | 9 | -2.39880075764678496228135351925E-2 |
| 8 | 5 | 1.87962962962962962962962962963E-2 | 15 | 10 | -1.65181846577159146362072750765E0 |
| 8 | 6 | 7.59713798041186071308940908624E-3 | 15 | 11 | 6.6245576312291875410038570643E-3 |
| 8 | 7 | -6.20331469003684620014386165793E-3 | 15 | 12 | 1.58274616357353025449630677489E0 |
| 9 | 1 | 7.51874092848346235045742434905E-2 | 15 | 13 | 1.41814358816879890444609368724E-1 |
| 9 | 3 | 3.08577739962730314518376303984E0 | 15 | 14 | 9.34364104078509623504023164074E-3 |
| 9 | 4 | 1.02777024339893729511937200674E0 | 16 | 1 | -1.72854883427149906480834787355E1 |
| 9 | 5 | -7.05175775924447499201004659796E-1 | 16 | 3 | -8.51259795469259804679437273731E0 |
| 9 | 6 | -2.78778538145998406742566524436E-2 | 16 | 4 | -2.80551782495675369923611987322E0 |
| 9 | 7 | 2.85805153481527740360777028619E-1 | 16 | 5 | 3.51919004458313930717487691522E1 |
| 9 | 8 | -2.99148657605355546429322500644E0 | 16 | 6 | 7.36701829077372132621075400047E-2 |
| 10 | 1 | 1.78548047149894440534834623505E-2 | 16 | 7 | -1.24584547068069360836645237480E0 |
| 10 | 3 | 5.16426091974317668823733047791E-3 | 16 | 8 | 6.34431710970183708662917963423E-1 |
| 10 | 4 | -3.43681317296491684696604004926E-3 | 16 | 9 | -2.99382194728825930237778557253E0 |
| 10 | 5 | 2.32372133623022296941501964109E-3 | 16 | 10 | 2.58384480400235896878583385841E1 |
| 10 | 6 | 5.14201281710644115584844336227E-4 | 16 | 11 | 7.60451062344283617975442262104E-1 |
| 10 | 7 | -4.33506924343872955710236185757E-4 | 16 | 12 | -2.72531371870126046405092063507E1 |
| 10 | 8 | -1.98666815536469802404438057068E-3 | 16 | 13 | -7.66905838615669874875007702991E0 |
| 11 | 1 | -1.18757288815909505564677978471E0 | 16 | 14 | -4.30717080246156504921654666097E-2 |
| 11 | 3 | 1.13239724969350324395658487817E-1 | 16 | 15 | 6.2429707127833603784354159720E0 |
| 11 | 4 | 4.15212324887617856117414455329E-2 | 17 | 1 | 2.38352607101136790911153507492E1 |
| 11 | 5 | 6.55529796136042083177053423077E-1 | 17 | 3 | 7.7738756188986200797865735259E0 |
| 11 | 6 | -1.43685337376884729223927086404E-3 | 17 | 4 | 2.57266456224187218683628421487E0 |
| 11 | 7 | 5.0120866241769657823310092071E-3 | 17 | 5 | -4.27301289332539628209504430154E1 |
| 11 | 8 | -7.07903479324387044990828600376E-1 | 17 | 6 | -6.84309506233548912384135730021E-2 |
| 11 | 9 | 2.87573843923931191319441255834E-1 | 17 | 7 | 1.30451149660509620846896301402E0 |
| 11 | 10 | 1.6607032033816552643102887011 5E0 | 17 | 8 | 3.34387872500134107823468680391E0 |
| 12 | 1 | 2.9326637612934925469041347559E-2 | 17 | 9 | 2.90875579291685107583165783656E0 |
| 12 | 3 | 2.27648379357749274981640086384E-1 | 17 | 10 | -3.51867845271413839756884852364E1 |
| 12 | 4 | 6.42188982905375335427504387651E-2 | 17 | 11 | -3.99562978143036844261012184204E-1 |
| 12 | 5 | 4.54015730358590118030630636463E-2 | 17 | 12 | 3.35175018689652853293607378584E1 |
| 12 | 6 | -7.94683657147007110737301019097E-4 | 17 | 13 | 9.93231523768212904432842366324E0 |
| 12 | 7 | 2.4861235148309752048563545005E-3 | 17 | 14 | 8.37781070877954481643354905414E-2 |
| 12 | 8 | -3.8188748663301078106113860815E-2 | 17 | 15 | -5.98649811509162584079618360502E0 |
| 12 | 9 | -6.35233112974332282906394142683E-4 | 17 | 16 | 9.88633847406948308075244574403E-2 |
| 12 | 10 | 3.05370536215106964984062651216E-2 | 18 | 1 | 3.07799270475412360737919532866E-2 |
| 13 | 1 | -6.0881392683555120739135947005E-2 | 18 | 8 | 9.27611792928056326667152622109E-2 |
| 13 | 3 | -4.43438802229705377088167804351E-3 | 18 | 9 | 1.22968704959599330688571533231E-2 |
| 13 | 4 | 4.85920262412870550959705396755E-2 | 18 | 11 | 1.54432519665121535509579924022E-1 |
| 13 | 5 | -1.23883926446739678566391896451E-1 | 18 | 12 | 2.62981836369090817847088871306E-1 |
| 13 | 6 | -5.40541062978150844511169211286E-3 | 18 | 13 | 1.08761305196919820431285749084E-1 |
| 13 | 7 | 2.22380334943109355360348696484E-3 | 18 | 14 | 5.88466616840491098220532404464E-3 |
| 13 | 8 | 3.9784387690238541442182630348E-2 | 18 | 15 | 2.6027080199888572877897873278E-1 |
| 13 | 9 | 4.00102754975348157308270345129E-4 | 18 | 16 | 4.29012449776118791099335675107E-2 |
| 13 | 10 | 1.39498241761820581940505774544E-1 | 18 | 17 | 2.89296487876585055315634624318E-2 |

Table 10: Continuation of the RKT9(7)8 from Table 9. Note that $b_i^*(\sigma) = \sum_{k=0}^{7} B_{ik}\sigma^k,\ i = 1, 2, \ldots, 18.$

| $i$ | $b_i$ | $i$ | $b_i$ |
|---|---|---|---|
| 1 | -1.2769053757776463532999815370 83E-1 | 12 | 1.1842135780322893846346410543 8E0 |
| 5 | -9.5E-1 | 13 | -1.5234603304815351279147533587 7E-1 |
| 7 | 2.5E-2 | 14 | 2.7921240846299173816160105515 9E-2 |
| 8 | 4.2096793303004495932229603967 6E-1 | 15 | -2.1386274865359161610767416330 3E-1 |
| 9 | 4.7333793027026717522224279053 3E-1 | 16 | 5.4348987998427705349577855513 6E-2 |
| 10 | 2.5E-1 | 17 | 5.1295209572488959529272526904 6E-2 |
| 11 | -4.3185560470425875675059336262 5E-2 | | |

| $i$ | $k$ | $B_{ik}$ | $i$ | $k$ | $B_{ik}$ |
|---|---|---|---|---|---|
| 1 | 0 | 1.0E0 | 12 | 1 | 1.153349972549467920396030963 15E1 |
| 1 | 1 | -2.777351168935506788994105773 23E1 | 12 | 2 | -1.188140838692033844056000726 43E2 |
| 1 | 2 | 2.516597886177936282989205449 6E2 | 12 | 3 | 3.081345675006919003878801211 03E2 |
| 1 | 3 | -1.086263919242813996028864022 21E3 | 12 | 4 | 2.334363520261893030751350075 82E1 |
| 1 | 4 | 2.533565399248855429402960458 55E3 | 12 | 5 | -9.667723851632948830721583194 07E2 |
| 1 | 5 | -3.242302501713999473080613111 67E3 | 12 | 6 | 1.157948870399966947017270868 52E3 |
| 1 | 6 | 2.130552849116231237801434893 76E3 | 12 | 7 | -4.151111219599050986210193190 93E2 |
| 1 | 7 | -5.604073244096642172678239137 03E2 | 13 | 1 | 2.156365421203432955254226177 98E0 |
| 5 | 1 | -1.186973648791990529608425057 05E1 | 13 | 2 | 2.230943763816378697264143408 83E1 |
| 5 | 2 | 1.024599154827099226654278356 03E2 | 13 | 3 | -2.148606916360788096278969967 08E2 |
| 5 | 3 | -1.160958382903468935568241629 57E2 | 13 | 4 | 6.776215723319894599665237938 88E2 |
| 5 | 4 | -6.706399765565724013298716951 75E2 | 13 | 5 | -1.014026697335017325349440370 2E3 |
| 5 | 5 | 1.944837215716690349249835548 86E3 | 13 | 6 | 7.310161539419173352796151040 43E2 |
| 5 | 6 | -1.854452307088306101833393034 96E3 | 13 | 7 | -2.041073790569809603762659055 44E2 |
| 5 | 7 | 6.057607272237450301009097592E2 | 14 | 1 | 5.088995866424825716604123143 83E-2 |
| 7 | 1 | 2.653835539316893650185647947 62E-1 | 14 | 2 | -7.422020879104591043589657397 67E-1 |
| 7 | 2 | -3.849572235338930825316767471 88E0 | 14 | 3 | 4.425958302284719289460039124 07E0 |
| 7 | 3 | 1.985834880863988183697897168 93E1 | 14 | 4 | -1.244652472798285315005748208 71E1 |
| 7 | 4 | -4.894258516148404650672966018 41E1 | 14 | 5 | 1.802057849099256683871186359 85E1 |
| 7 | 5 | 6.256800780344131701032411139 52E1 | 14 | 6 | -1.295266799026140611843521352 96E1 |
| 7 | 6 | -4.008609550388550391190268040 46E1 | 14 | 7 | 3.649852720381588898495922726 44E0 |
| 7 | 7 | 1.018651273469559303162746018 13E1 | 15 | 1 | -4.430014721452922669327941949 54E0 |
| 8 | 1 | 4.108816885949314465612832609 61E-1 | 15 | 2 | 6.215785602709654552836326558 49E1 |
| 8 | 2 | 3.120200010326628151023996496 94E1 | 15 | 3 | -3.076768803644281055772959473 94E2 |
| 8 | 3 | -2.497077013126749896595984348 86E2 | 15 | 4 | 7.151287135953468518643962765 83E2 |
| 8 | 4 | 7.781955892105118253954852147 43E2 | 15 | 5 | -8.419015628018948565704020382 32E2 |
| 8 | 5 | -1.172010779205192079600274746 54E3 | 15 | 6 | 4.869974808846874740611821899 24E2 |
| 8 | 6 | 8.505323948159900904832537860 76E2 | 15 | 7 | -1.100153218173561009081368257 84E2 |
| 8 | 7 | -2.385296241212032539430003523 64E2 | 16 | 1 | 6.397463913501998295056670868 82E-1 |
| 9 | 1 | 4.149066899839339140893541231 83E0 | 16 | 2 | -8.924473311363209933147282743 81E0 |
| 9 | 2 | -6.020108301221202912253226298 4E1 | 16 | 3 | 4.393759772736324525802119110 17E1 |
| 9 | 3 | 3.134025645069361112401331590 1E2 | 16 | 4 | -1.000553975228759815920036369 38E2 |
| 9 | 4 | -7.819118866721812477202310545 71E2 | 16 | 5 | 1.132235008686914695409044281 59E2 |
| 9 | 5 | 1.012497750807176391848212814 49E3 | 16 | 6 | -6.090410190967222961630083017 21E1 |
| 9 | 6 | -6.566607109995616952644141765 54E2 | 16 | 7 | 1.212602900148411839213039707 42E1 |
| 9 | 7 | 1.687365953404990898110068365 3E2 | 17 | 1 | -3.429365185659148434454233400 05E-2 |
| 10 | 1 | 2.700879589742588805147960953 06E1 | 17 | 2 | 1.022639927895100825226825844 94E0 |
| 10 | 2 | -3.083001874669668628915170919 84E2 | 17 | 3 | -7.675530827321086176441538088 83E0 |
| 10 | 3 | 1.446551779480422798514288140 73E3 | 17 | 4 | 2.876827077348899809172989562 23E1 |
| 10 | 4 | -3.517001741103875714561727323 56E3 | 17 | 5 | -5.453047001300680823812373922 45E1 |
| 10 | 5 | 4.608416570490027512245788026 49E3 | 17 | 6 | 4.878225047679627973100957641 61E1 |
| 10 | 6 | -3.072586873639840088714838842 47E3 | 17 | 7 | -1.630393703720823424352491477 36E1 |
| 10 | 7 | 8.159116563428064673565274812 71E2 | 18 | 1 | 4.507931010337410128430426794 7E-2 |
| 11 | 1 | -2.152152296023295011425754627 52E0 | 18 | 2 | -1.399710372327775815051348352 54E0 |
| 11 | 2 | 3.141967455839738629670392086 79E1 | 18 | 3 | 1.129959638924927475233079932 72E1 |
| 11 | 3 | -1.653298510419240506521713198 4E2 | 18 | 4 | -4.403937366814883358687545958 51E1 |
| 11 | 4 | 4.184143050503095834188871719 55E2 | 18 | 5 | 8.563835383094499285293334099 09E1 |
| 11 | 5 | -5.536575817755591736756978087 08E2 | 18 | 6 | -7.862889621342194948701258438 02E1 |
| 11 | 6 | 3.704416537093596105725309437 29E2 | 18 | 7 | 2.708495072360091718239094773 18E1 |
| 11 | 7 | -9.898161568489493941331757345 26E1 | | | |

## 2.7 Order 9

From Table 1 there are 486 equations associated with a process of order 9 and the value of $C_9$ is not precisely known. This section briefly outlines a solution of the equations with $s^* = 22$. The solution builds upon the analysis from the previous orders and assumes the FSAL conditions (2) together with the equations (8), (10), (11), (14), (16), (17) and the following additional requirements

$$b_5^* = b_7^* = b_{10}^* = Q_{i4} = 0, \ i = 8, 9, 11, \ldots, s^*, \tag{18}$$

which imply $c_6 = 7c_3/15$, $c_9 = 7c_3/3$ and $c_{10}$ is restricted as follows

$$c_{10} = \frac{8pc_3(329c_3^2 - 175sc_3 + 109p)}{4557c_3^4 - 3878sc_3^3 + (553s^2 + 3034p)c_3^2 - 790spc_3 + 237p^2}$$

$$\text{where} \quad s = c_{11} + c_{12} \quad \text{and} \quad p = c_{11}c_{12}.$$

It is now possible by imposing equation (12) to make the discrete formula of order 10 but not of order 11 without violating equation (5). The remaining degrees of freedom ($c_3$, $c_i, i = 11, 12, \ldots, 20$) can now be chosen to give small values for $J^{(10)}(\approx 2.86 \times 10^{-7})$ and $A^{(11)}(\approx 7.32 \times 10^{-9})$. The resulting triple utilising an eighth order embedded process can be found in Tables 11, 12 and 13 which contain only the non-zero entities.

Table 11: RKT10(8)9 which is continued in Table 12. Note that $\widehat{b}_i = a_{22i}$, $i = 1, 2, \ldots, 21$ and $c_i = \sum_{j=1}^{i-1} a_{ij}$, $i = 1, 2, \ldots, 22$.

| $i$ | $j$ | $a_{ij}$ | $i$ | $j$ | $a_{ij}$ |
|---|---|---|---|---|---|
| 2 | 1 | 1.7142857142857142857142857142857429E-1 | 14 | 4 | 7.61153218749380388515161951786E-3 |
| 3 | 1 | 6.4285714285714285714285714285 7E-2 | 14 | 5 | -1.4139403105494904804162178916 7E-2 |
| 3 | 2 | 1.92857142857142857142857142857E-1 | 14 | 6 | -1.7095555323508155354294620146E-1 |
| 4 | 1 | 2.57142857142857142857142857143E-1 | 14 | 7 | 5.30514733461304180522911933357E-1 |
| 4 | 2 | -7.71428571428571428571428571429E-1 | 14 | 8 | -5.11031289248798174898569773741E-2 |
| 4 | 3 | 1.02857142857142857142857142857E0 | 14 | 9 | 9.40085414679690751975117567394E-3 |
| 5 | 1 | 8.57142857142857142857142857143E-2 | 14 | 10 | -4.52042349903216396534832819473E-2 |
| 5 | 3 | 3.42857142857142857142857142857E-1 | 14 | 11 | -9.95786229376399381069676911848E-3 |
| 5 | 4 | 8.57142857142857142857142857143E-2 | 14 | 12 | -2.21770771247996610673849359841E-1 |
| 6 | 1 | 6.80444444444444444444444444445E-2 | 14 | 13 | 6.09529832932070843559292477197E-3 |
| 6 | 2 | 6.44E-2 | 15 | 1 | 2.71722196517267073721919988737E-2 |
| 6 | 3 | -9.95555555555555555555555555555E-3 | 15 | 3 | 3.36111714500765050195471053533E-3 |
| 6 | 4 | -1.39377777777777777777777777778E-2 | 15 | 4 | 2.92271056087621782778670481333E-4 |
| 6 | 5 | 1.14488888888888888888888888889E-2 | 15 | 5 | -9.10415966436935539116562416401E-4 |
| 7 | 1 | 4.70888888888888888888888888889E-2 | 15 | 6 | -6.56442866656426475295926935115E-3 |
| 7 | 3 | -1.44277777777777777777777777778E-2 | 15 | 7 | 3.30757277239804600985460593148E-2 |
| 7 | 4 | 3.59333333333333333333333333333E-3 | 15 | 8 | 7.24437088394392664852811821195E-2 |
| 7 | 5 | -2.50444444444444444444444444444E-3 | 15 | 9 | -9.5131112062075069280861431775E-4 |
| 7 | 6 | 8.625E-2 | 15 | 10 | -1.02426896941633891961563238252E-1 |
| 8 | 1 | 3.82653061224489795918367346939E-2 | 15 | 11 | 1.60643797941239012292483444089E-3 |
| 8 | 3 | 5.08928571428571428571428571429E-2 | 15 | 12 | -5.51368718332946785146311721738E-3 |
| 8 | 4 | 4.42546583850931677018633540373E-3 | 15 | 13 | -9.96109320699920624283449061858E-4 |
| 8 | 5 | -4.89130434782608695652173913044E-3 | 15 | 14 | 1.94113668036311350585167948512E-2 |
| 8 | 6 | -9.93962768742387584253253016032E-2 | 16 | 1 | 2.53806341201199324666315214607E-2 |
| 8 | 7 | 2.67846809261106549019823970636E-1 | 16 | 3 | 3.7531000076108783558977100526E-1 |
| 9 | 1 | 5.27777777777777777777777777778E-2 | 16 | 4 | 3.26356522400945943991105221965E-2 |
| 9 | 3 | 7.77139450056116722783389450056E-1 | 16 | 5 | 5.26325269986361373786615121734E-2 |
| 9 | 4 | 6.75773434831405845898599521788E-2 | 16 | 6 | -7.32999065951547844478691298224E-1 |
| 9 | 5 | 1.39548260381593714927048260382E-1 | 16 | 7 | 9.67678996523823578769532202087E-1 |
| 9 | 6 | -1.51779193160336827633614023028E0 | 16 | 8 | 2.15791589050093115948614702104E-1 |
| 9 | 7 | 1.6196941055164117545970097955E0 | 16 | 9 | 3.67553527064403441681259310579E-2 |
| 9 | 8 | -5.38945005611672278338945005612E-1 | 16 | 10 | -1.81530947002862058971722916931E-1 |
| 10 | 1 | 3.88353461879046773963440614956E-2 | 16 | 11 | -4.09922890594496687340524305813E-2 |
| 10 | 3 | 4.79464320723202865567932468526E-2 | 16 | 12 | 2.15236785669238150386436039164E-1 |
| 10 | 4 | 4.5769292181296043927159035332E-3 | 16 | 13 | 2.2953702084828654883891671399E-2 |
| 10 | 5 | -4.88880091474728780149720288497E-3 | 16 | 14 | -4.55519604807169438472975127833E-1 |
| 10 | 6 | -8.87583529719000069454414466987E-2 | 17 | 1 | 1.51078938700081213900635874424E-3 |
| 10 | 7 | 2.54437297141552907740401224724E-1 | 17 | 3 | 7.53566292007116950326864746808E-1 |
| 10 | 8 | -1.83774831423237984479567378777E-2 | 17 | 4 | 6.55275036527927782892925866789E-2 |
| 11 | 1 | -6.6675065650641000724509489908E-2 | 17 | 5 | 8.46747951362504290540732821269E-2 |
| 11 | 3 | -1.05367308433973749664957424838E0 | 17 | 6 | -1.47175238350604897998910257348E0 |
| 11 | 4 | -9.16237464643249997086586302935E-2 | 17 | 7 | 2.1544076679551517246754174083E0 |
| 11 | 5 | -2.43751186211772783274461089616E-1 | 17 | 8 | -2.95817130713649583710180908561E-1 |
| 11 | 6 | 2.05787584949265887766230841109E0 | 17 | 9 | 1.41108557295954372127216044522E-1 |
| 11 | 7 | -1.1231886312992026938265345702E0 | 17 | 10 | 9.93363998637091884807695896669E-4 |
| 11 | 8 | 6.08712534525018370495459105367E0 | 17 | 11 | -9.49402391923509994190801602882E-2 |
| 11 | 9 | 4.3459596156342027900897900295E-1 | 17 | 12 | -4.09994248367412059381729023618E-1 |
| 11 | 10 | -5.27847077920393856587197675801E0 | 17 | 13 | 5.17440541722880221140981310637E-2 |
| 12 | 1 | 3.87942523461117041377971875372E-2 | 17 | 14 | -3.62816777296553733507007891428E-1 |
| 12 | 3 | 2.62854887215373583030498361145E-2 | 17 | 15 | 6.5360732511345784837404560211E-2 |
| 12 | 4 | 2.28569467143803115678694227083E-3 | 17 | 16 | -1.83572977040522609441080256973E-1 |
| 12 | 5 | -5.22163450008847739571621732795E-3 | 18 | 1 | 5.33868129054430002753837028879E-3 |
| 12 | 6 | -5.13368645703417601208247348983E-2 | 18 | 3 | 1.47470382943225743303202666729E-1 |
| 12 | 7 | 1.85631908317522880760632258473E-1 | 18 | 4 | 1.28235115602804994176697971069E-2 |
| 12 | 8 | 1.20451382273352635107082214494E-1 | 18 | 5 | -1.01268353155170256618697863596E-2 |
| 12 | 9 | 9.09566403893770608709777784378E-4 | 18 | 6 | -2.88016979919787170816119555695E-1 |
| 12 | 10 | -1.92799793663426142557517264448E-1 | 18 | 7 | 6.9824510039379039160059721042E-1 |
| 13 | 1 | -6.12172589287871747048472015259E-3 | 18 | 8 | -3.37974521430012528505134988237E-1 |
| 13 | 3 | -2.35054787000214690914783867723E-1 | 18 | 9 | 2.86414411975806736110165973921E-2 |
| 13 | 4 | -2.04395466956708426882420754542E-2 | 18 | 10 | 3.7131816347799479039918797965E-1 |
| 13 | 5 | -4.82769557359166681354132197148E-2 | 18 | 11 | -2.44811592512287236385020747211E-2 |
| 13 | 6 | 4.59073669684266375817327005265E-1 | 18 | 12 | -3.4196135976548130845457690019E-1 |
| 13 | 7 | -1.92474861837713007220976018487E-1 | 18 | 13 | 1.42807674655094881558598522546E-2 |
| 13 | 8 | 2.91535209036744365879418079661E0 | 18 | 14 | -5.3315265057770455656362956804E-2 |
| 13 | 9 | 2.53783494031343851135063073233E-1 | 18 | 15 | 7.75697896089233534885598800183E-2 |
| 13 | 10 | -2.70463757085476242661593215442E0 | 18 | 16 | -1.56043051859614787897656163996E-2 |
| 13 | 11 | 5.73922969626822774434055704329E-2 | 18 | 17 | -6.42963423431277070352269767695E-3 |
| 13 | 12 | 2.71403896971420189855855610407E-1 | 19 | 1 | 2.14923549969032701447324062889E-1 |
| 14 | 1 | 3.86425821831108415990101575487E-2 | 19 | 3 | -3.31284355637506491687356054727E-2 |
| 14 | 3 | 8.75326201561787446792436244553E-2 | 19 | 4 | -2.88073352728266514510744395414E-3 |

Table 12: Continuation of the RKT10(8)9 from Table 11

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| $i$ | $j$ | $a_{ij}$ | $i$ | $j$ | $a_{ij}$ |
|---|---|---|---|---|---|
| 19 | 5 | -1.24468160743674116046379662049E-2 | 20 | 19 | 2.39771105500080177790811432148E0 |
| 19 | 6 | 6.47014795113952630750063665388E-2 | 21 | 1 | 4.13666331788568646314671459781E-2 |
| 19 | 7 | 5.14370598499741879276769473802E-3 | 21 | 3 | -4.6085087722059010473154473953E-1 |
| 19 | 8 | -1.43046105116307215521975907197E0 | 21 | 4 | -4.00739893235295743244821512635E-2 |
| 19 | 9 | 2.8721437377474197463041334418E-2 | 21 | 5 | -1.87482210748488705850730209348E-1 |
| 19 | 10 | 8.78687940562103891093510434214E-3 | 21 | 6 | 9.00064644855228349679196498542E-1 |
| 19 | 11 | -1.57537511615043694291339630709E-1 | 21 | 7 | 6.00984146321174224002305454373E-1 |
| 19 | 12 | 1.58483196482085928507966531199E0 | 21 | 8 | 1.684608594095337611555270728E0 |
| 19 | 13 | 2.4923172103039147318147293828E-1 | 21 | 9 | 5.30003044813386314235741018963E0 |
| 19 | 14 | -1.30642815150584000544713644415E0 | 21 | 10 | -4.2080611537944712705604048092E0 |
| 19 | 15 | -5.28959072870432651573430444804E-1 | 21 | 11 | 4.32979699798406374581760208508E0 |
| 19 | 16 | 1.00741207592023609430941633181E0 | 21 | 12 | 1.95388343086605943251642605032E0 |
| 19 | 17 | -8.775590457123529971261338772E-1 | 21 | 13 | -7.74274861680373310963503498563E0 |
| 19 | 18 | 1.99675911512324586842776245057E0 | 21 | 14 | -3.13510062968838311211920910556E0 |
| 20 | 1 | -2.97166149971555334413883837806E-2 | 21 | 15 | -4.78663380485811935550439247685E-1 |
| 20 | 3 | -3.93561746825012462479575933126E-1 | 21 | 16 | -6.03141603406035774837148626753E0 |
| 20 | 4 | -3.42227605934793445634413854892E-2 | 21 | 17 | 1.35672452470896311806863413262E0 |
| 20 | 5 | -1.58696967693592826136419752921E-1 | 21 | 18 | 4.36880260572859992367639746079E0 |
| 20 | 6 | 7.68645632229322013542244960244E-1 | 21 | 19 | 2.75280494424409090789700873245E0 |
| 20 | 7 | 4.96090687123684429260132170967E-1 | 21 | 20 | -4.67007799087175905838696203928E-3 |
| 20 | 8 | 1.92649797393520230726050104946E0 | 22 | 1 | 3.92740785119914854000268988182E-2 |
| 20 | 9 | 4.46513079425448299443002782977E0 | 22 | 12 | 4.68444158055361823794214145044E-1 |
| 20 | 10 | -3.58323257191568143688861876877E0 | 22 | 13 | -2.3636350361826723946475816494E-2 |
| 20 | 11 | 3.7003765504475496211886812753E0 | 22 | 14 | -4.10019740322940585768982972727E-1 |
| 20 | 12 | 1.24038802088878265649236798944E0 | 22 | 15 | -2.97256966740860193040952924122E-2 |
| 20 | 13 | -6.61304590135240072826990058323E0 | 22 | 16 | 5.76476584209117916904648266301E-1 |
| 20 | 14 | -2.32192461516956715430435503205E0 | 22 | 17 | -3.20051029759055386286764973125E-1 |
| 20 | 15 | -2.08786912217290485002957585106E-1 | 22 | 18 | 3.73265122088390393431398983892E-1 |
| 20 | 16 | -5.1100351397439630950486041015E0 | 22 | 19 | 2.46074190640257467948091174643E-1 |
| 20 | 17 | 1.25420454344261812247625210394E0 | 22 | 20 | 4.88504071988586166889784770355E-1 |
| 20 | 18 | 3.20027172318569914357693982537E0 | 22 | 21 | -4.08605388375796539061845184294E-1 |

| $i$ | $b_i$ | $i$ | $b_i$ |
|---|---|---|---|
| 1 | 1.6923713048071774678167962342E-1 | 15 | -4.11056665125909373103182261953E-1 |
| 8 | -5.5E-1 | 16 | 4.20137072018899352803788919616E-1 |
| 9 | 2.0E-1 | 17 | -4.07267073390825120527856167596E-1 |
| 11 | 1.0E-1 | 18 | 1.14415870529411131962321567601E0 |
| 12 | 1.49788508979634065630064615597E0 | 19 | 3.00298070359876651423408830867E-1 |
| 13 | -2.12348185097986841873295284576E-1 | 20 | 4.99331001586965891752992972611E-1 |
| 14 | -1.33059869039202370169631358297E0 | 21 | -4.197764555301665814850848814E-1 |

Table 13: Continuation of the RKT10(8)9 from Tables 11 and 12. Note that $b_i^*(\sigma) = \sum_{k=0}^{8} B_{ik}\sigma^k,\ i = 1, 2, \ldots, 22.$

| $i$ | $k$ | $B_{ik}$ | $i$ | $k$ | $B_{ik}$ |
|---|---|---|---|---|---|
| 1 | 0 | 1.0E0 | 15 | 5 | 7.16249850677466769812869812474E3 |
| 1 | 1 | -2.06297782812448935889003344791E1 | 15 | 6 | -6.09824643516338291055945318713E3 |
| 1 | 2 | 1.99970176048388561019274681466E2 | 15 | 7 | 2.81520634280299703935993919373E3 |
| 1 | 3 | -9.22654837092107700126634271173E2 | 15 | 8 | -5.48241131264206537738512642553E2 |
| 1 | 4 | 2.31222312792268813253542444857E3 | 16 | 1 | -3.01648978960918371803963335019E0 |
| 1 | 5 | -3.36805982998121579505655492833E3 | 16 | 2 | 5.87536701899315845616874210878E1 |
| 1 | 6 | 2.86779930979446876870346996672E3 | 16 | 3 | -5.55607498535208412476473482541E2 |
| 1 | 7 | -1.33009659749051336937054709224E3 | 16 | 4 | 2.39215015269716575399237428541E3 |
| 1 | 8 | 2.60487703158048287369867556359E2 | 16 | 5 | -5.13695842994462016499807820042E3 |
| 8 | 1 | -4.58615459099727348406065425338E0 | 16 | 6 | 5.82660273225751977620329263459E3 |
| 8 | 2 | -3.43246388036553518122971208411E1 | 16 | 7 | -3.35911159614824317109118564421E3 |
| 8 | 3 | 3.09486850078952618787463125081E2 | 16 | 8 | 7.77763935857272935443327271475E2 |
| 8 | 4 | -6.14802040674386044371724189846E2 | 17 | 1 | -1.56065665159331057732607100341E0 |
| 8 | 5 | 1.73108038323864389835096306022E2 | 17 | 2 | 1.39173777524727035508035241556E1 |
| 8 | 6 | 6.59677271084104987273220731019E2 | 17 | 3 | -1.33437194028282508797164633795E2 |
| 8 | 7 | -6.88853829878109035239941941896E2 | 17 | 4 | 6.73365409739643774512539517957E2 |
| 8 | 8 | 2.00294504460225709012243744517E2 | 17 | 5 | -1.67196796583630511285236447147E3 |
| 9 | 1 | 3.47732466883096028887251751081E0 | 17 | 6 | 2.11584335238948234994960839082E3 |
| 9 | 2 | -5.92490712105459256032488058304E1 | 17 | 7 | -1.31751860530872646114657780275E3 |
| 9 | 3 | 5.68658548319989900493945781598E2 | 17 | 8 | 3.21038230913549509974194781111E2 |
| 9 | 4 | -2.56306468722100318797831395987E3 | 18 | 1 | 9.34193888791873862812244205862E0 |
| 9 | 5 | 5.80470372175372051326344179587E3 | 18 | 2 | -3.87753944045788759746877558967E1 |
| 9 | 6 | -6.91311865221701067191638950388E3 | 18 | 3 | 3.33501822703837688986890042108E2 |
| 9 | 7 | 4.15224530103838188508336576403E3 | 18 | 4 | -1.89106977539173495698057758062E3 |
| 9 | 8 | -9.93652485132363473631673589432E2 | 18 | 5 | 4.82013028954164025673410278997E3 |
| 11 | 1 | 1.34663686046235080290387565235E0 | 18 | 6 | -6.02334485547518422386023637806E3 |
| 11 | 2 | -2.58804873304224303664083918462E1 | 18 | 7 | 3.65368701068403665949817652999E3 |
| 11 | 3 | 2.48212680978950604447262596116E2 | 18 | 8 | -8.63097771423846896638358690559E2 |
| 11 | 4 | -1.11476732055076869932396566606E3 | 19 | 1 | -1.91132340938196948061289881587E-1 |
| 11 | 5 | 2.54147893462516058275681867833E3 | 19 | 2 | 1.94583814910815996810661656457E0 |
| 11 | 6 | -3.06590343710225795141536267335E3 | 19 | 3 | -1.91764522790416307850488506448E1 |
| 11 | 7 | 1.87123241359665405619794221169E3 | 19 | 4 | 8.24223992823041297141088673633E1 |
| 11 | 8 | -4.55719421077778513099190635734E2 | 19 | 5 | -1.73932819187320347159046403168E2 |
| 12 | 1 | -1.09848345718521052705684043776E-1 | 19 | 6 | 1.95129106419493159569653996863E2 |
| 12 | 2 | 2.37545383059934117890189125521E2 | 19 | 7 | -1.10389525803352906554037850095E2 |
| 12 | 3 | -1.48031852859346556766872317475E3 | 19 | 8 | 2.44386599503878896622730041735E1 |
| 12 | 4 | 3.73732991166234759962890853683E3 | 20 | 1 | 2.83986897250454500650248073256E0 |
| 12 | 5 | -4.84230983703745358888134561558E3 | 20 | 2 | -4.91430010822156343410779466753E1 |
| 12 | 6 | 3.38228426572331858440361176138E3 | 20 | 3 | 4.85603254086018436823082501704E2 |
| 12 | 7 | -1.20565338653341576589325198712E3 | 20 | 4 | -2.27195016307781611626194920667E3 |
| 12 | 8 | 1.71700484222508503397111251905E2 | 20 | 5 | 5.45896500681048982559906085068E3 |
| 13 | 1 | -1.70630269533425952944445664144E0 | 20 | 6 | -7.02973017700117520597832555491E3 |
| 13 | 2 | 3.34627541102967460124100531515E1 | 20 | 7 | 4.62172517578571248259481125538E3 |
| 13 | 3 | -3.23243454145416293380601322224E2 | 20 | 8 | -1.21782146042152974727521459389E3 |
| 13 | 4 | 1.47881306032917864920140996938E3 | 21 | 1 | -2.58317537250990449928827222222E0 |
| 13 | 5 | -3.44623375671007767107644058027E3 | 21 | 2 | 4.50229147599410572355643033391E1 |
| 13 | 6 | 4.24829425145108678784311765986E3 | 21 | 3 | -4.42245832045825204341093428442E2 |
| 13 | 7 | -2.64596733630927079224341425009E3 | 21 | 4 | 2.05450590783890414371356509087E3 |
| 13 | 8 | 6.56557147619175006449016451009E2 | 21 | 5 | -4.90444245053620048359003799753E3 |
| 14 | 1 | -7.78853771367157794303769750711E0 | 21 | 6 | 6.27850727502860920633477140288E3 |
| 14 | 2 | -2.27081279057536740907348566 8E1 | 21 | 7 | -4.10621437902256669581814655 56E3 |
| 14 | 3 | 8.09079339772872291769019776807E1 | 21 | 8 | 1.07704113396127208442560361151E3 |
| 14 | 4 | 4.94706154179900746072182756 83E2 | 22 | 1 | -9.50046718078759926599952795 6E-2 |
| 14 | 5 | -2.16269478733290895408424222346E3 | 22 | 2 | 1.36456304182752713536970146404E0 |
| 14 | 6 | 3.18669482783876005542919399735E3 | 22 | 3 | -1.62187843007174977273779256203E1 |
| 14 | 7 | -2.08159122651751575390515006417E3 | 22 | 4 | 9.12757176743336247657360025741E1 |
| 14 | 8 | 5.12063743733578988759117126981E2 | 22 | 5 | -2.54284621263441148619108121807E2 |
| 15 | 1 | 2.52613110637084026071227727079E1 | 22 | 6 | 3.69511164972167288019826755834E2 |
| 15 | 2 | -3.61901956374728565184950548 98E2 | 22 | 7 | -2.68699760896068171471981770269E2 |
| 15 | 3 | 1.86653149087502833658757106 49E3 | 22 | 8 | 7.71467254437062538901953531036E1 |
| 15 | 4 | -4.86113785441075754921971887272E3 | | | |

# 3 Numerical Testing and Comparisons

A representative sample of problem equations taken from the DETEST suite [10] and from those used by Hairer et al. [9] was used for the testing of the various triples. Standard parameters were used for all the triples involved and relative error per step was monitored for a range of tolerances between $10^{-3}$ and $10^{-14}$. For those problems where the true solution was known $M$, $M^*$ and $M_E$ were computed where $M$ is the maximum absolute value of $\boldsymbol{\varepsilon}_n$ over all equation components and all integration steps, $M^*$ is the maximum absolute value of $\boldsymbol{\varepsilon}_{n+\sigma}$ over all equation components and over 100 equally spaced points within each step interval $[x_n, x_{n+1}]$ and $M_E$ is the maximum absolute value of the discrete global error, $\boldsymbol{\varepsilon}(x = xend)$, over all equation components at the integration end point. Efficiency plots of $log_{10}M/M^*/M_E$ against function evaluations could then be considered. If the true solution was not available approximate values of the global errors were obtained by solving the equation system with a higher order RK process with suitable tolerance and quadruple precision arithmetic.

For the triples of order 3 the practical tests over a wide range of problems confirm, as expected from the sizes of the $A^{(4)}$ values, that there is little to choose between the RKT3(2)3 and the corresponding triple of Owren and Zennaro (RK3(2)OZ) except where the absence of the endpoint evaluation in the error estimation process is significant. To demonstrate this situation consideration is given to the EULR problem of [9] with a modified exterior function of $(1/4)cos^2 x sin x$ instead of $(1/4)sin^2 x$. Figure 1 shows the plots of $log_{10}M_E$ $(x_{end} = 20)$ against function evaluations.

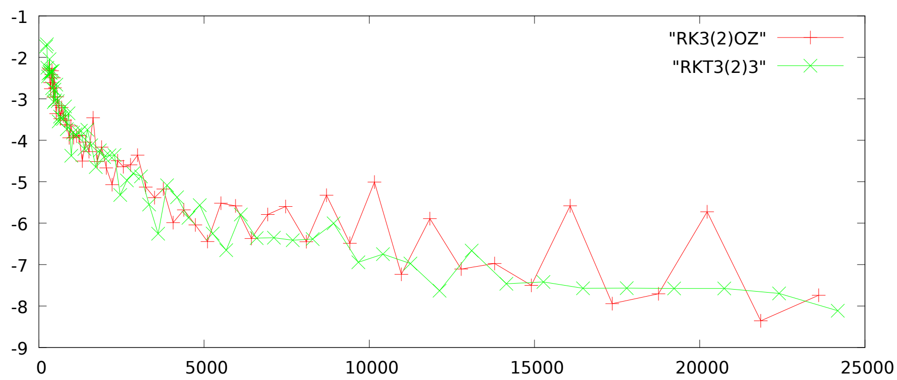


Figure 1: Modified EULR problem - $log_{10}M_E$ against Function Evaluations.

For the triples of order 4 the practical tests show that, as expected from the sizes of the $A^{(5)}$ values, the RKT4(3)4 generally gives significantly better results when considering $M$, $M^*$ or $M_E$ than the corresponding Owren and Zennaro triple (RK4(3)OZ). Typical situations are the solution of DETEST problems D3 and A4 [10] and the AREN and BRUS problems of [9] as seen in Figures 2 to 5. Problem A4 was chosen since this does show significant differences between plots using $M$ and $M^*$ for each of the two triples. On many problems there is no significant difference between the $M$ and $M^*$ plots for the same triple. As expected the RK4(3)OZ gives similar poor results to the RK3(2)OZ on the modified EULR problem because of the absence of the endpoint evaluation in the error estimation process.

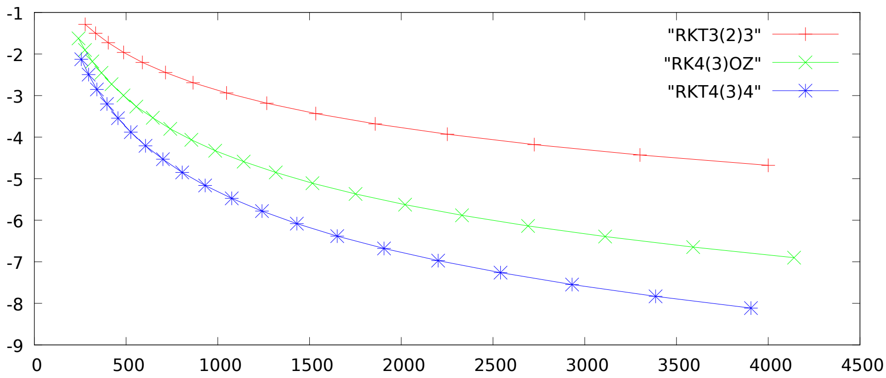


Figure 2: DETEST Problem D3 - $log_{10}M$ against Function Evaluations.

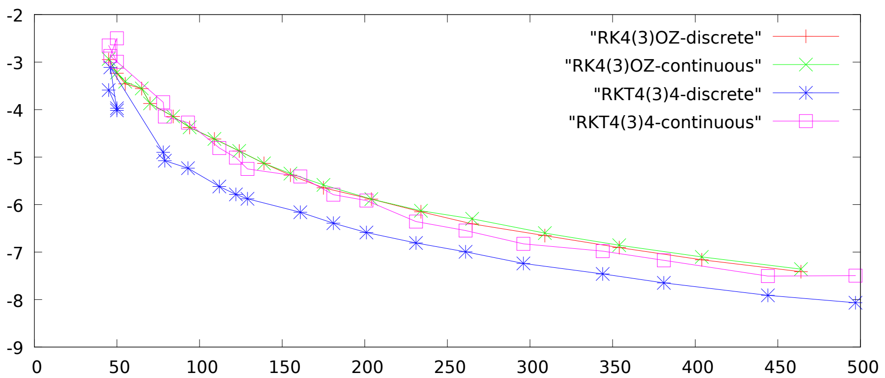


Figure 3: DETEST Problem A4 - $log_{10}M/M^*$ against Function Evaluations.

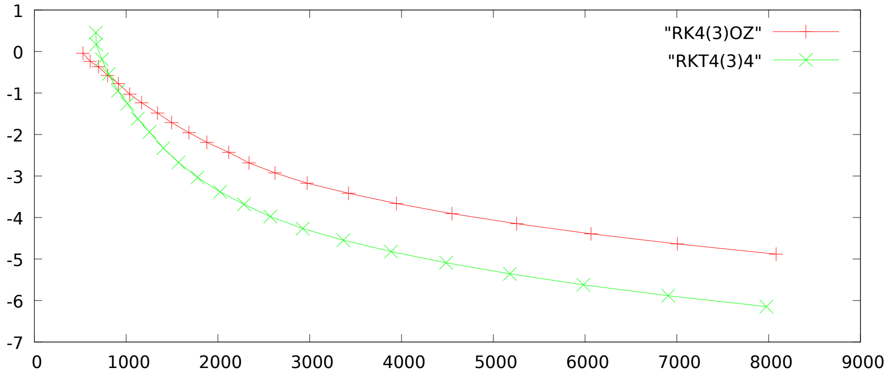


Figure 4: AREN Problem - $log_{10}M$ against Function Evaluations.

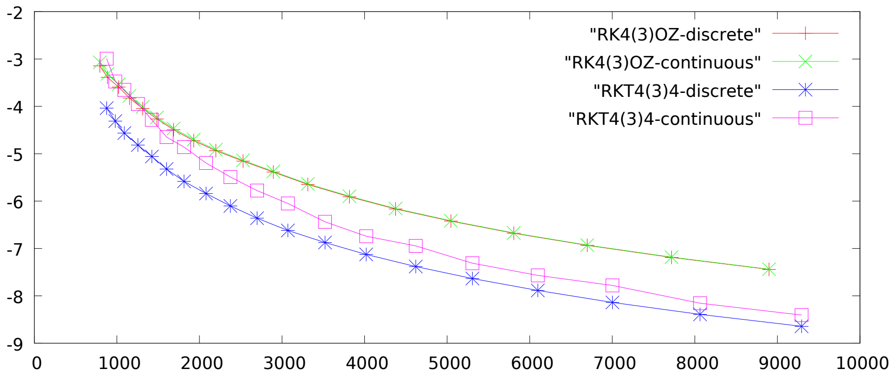


Figure 5: BRUS Problem - $log_{10}M/M^*$ against Function Evaluations.

For order 5 the practical tests show that the RKT5(4)5 is preferable to the triple of Owren and Zennaro (RK5(4)OZ) but neither is as efficient as the RK5(4)7FM with the fourth order dense formula on the problems tested. For $log_{10}M$ against function evaluations the RK5(4)7FM was only bettered by the RKT5(4)5 on two of the problems out of the dozen or so tested and was never bettered by the RK5(4)OZ which gave extremely poor results on the modified EULR problem and yet surprisingly good results on the BRUS problem. For $log_{10}M^*$ against function evaluations the RK5(4)7FM with the fourth order dense formula was still generally more efficient than the other two triples. Use of the fifth order continuous formula with the RK5(4)7FM was rarely worth the extra cost of two evaluations per step. Figures 6 to 10 illustrate the situation.

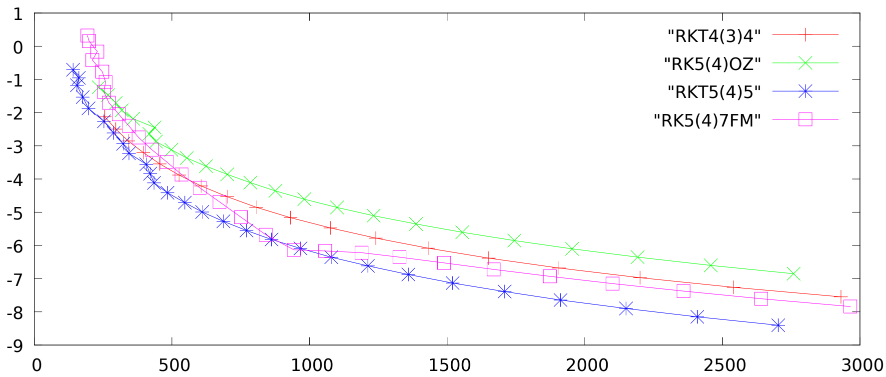


Figure 6: DETEST Problem D3 - $log_{10}M$ against Function Evaluations.

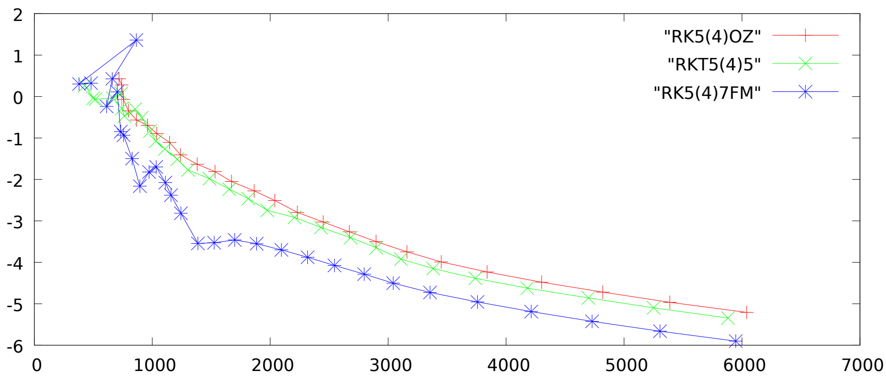


Figure 7: AREN Problem - $log_{10}M$ against Function Evaluations.

For order 6 the practical tests confirm that there is a preference for the RKT7(5)6 against the RK6(5)9FM with the fifth order continuous formula particularly at the more stringent tolerances. The use of the sixth order continuous process with the RK6(5)9FM did not appear to be worth the extra cost. Figures 11 to 13 illustrate the position.

For orders 7 and above the practical tests indicate that the RKT8(6)7, the RK8(6)12M or the RK8(7)13M are not as efficient as either the RKT9(7)8 or the RKT10(8)9. Figures 14 to 19 compare these with the RK5(4)7FM and the RKT7(5)6. It should be noted that the size of some of the RK coefficients in the higher order triples means that it is advisable to use high precision arithmetic when using stringent tolerances.

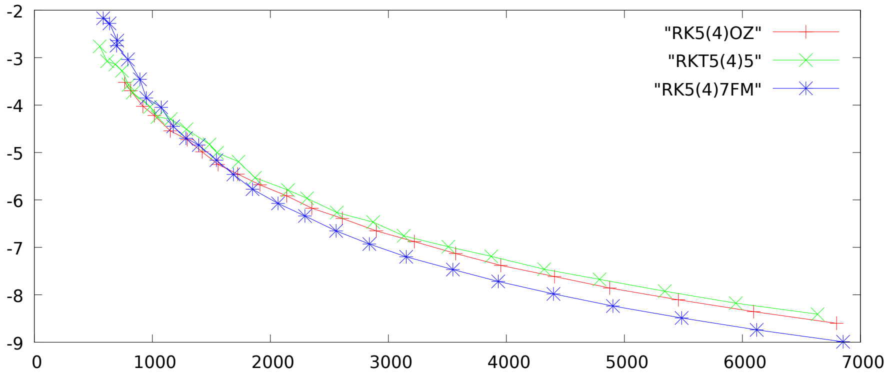


Figure 8: BRUS Problem - $log_{10}M$ against Function Evaluations.

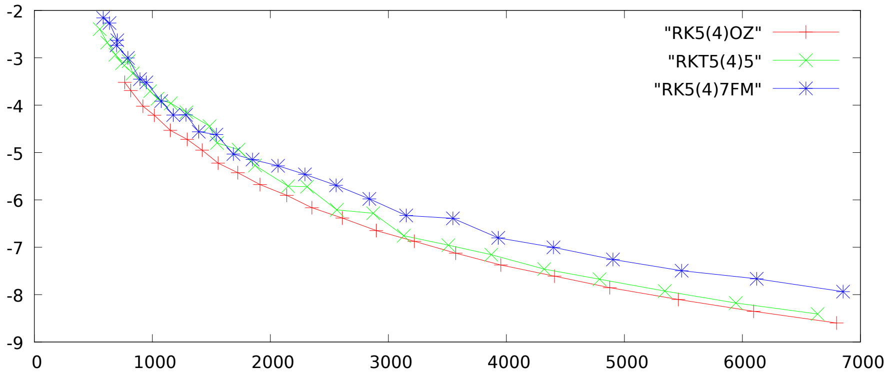


Figure 9: BRUS Problem - $log_{10}M^*$ against Function Evaluations.

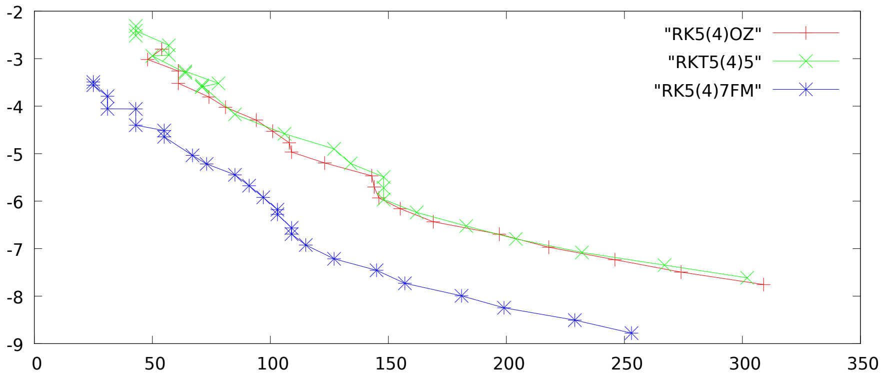


Figure 10: DETEST Problem E5 - $log_{10}M$ against Function Evaluations.

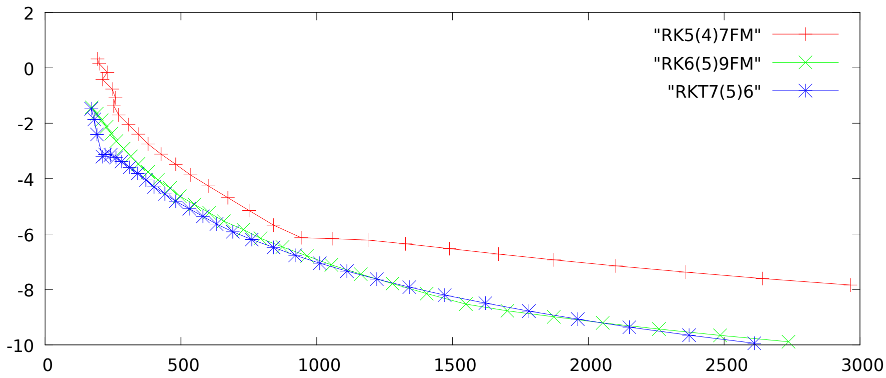


Figure 11: DETEST Problem D3 - $log_{10}M$ against Function Evaluations.

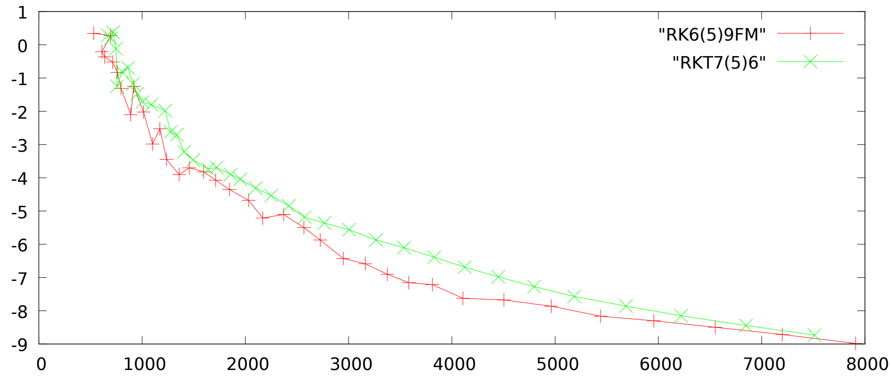


Figure 12: AREN Problem - $log_{10}M$ against Function Evaluations.

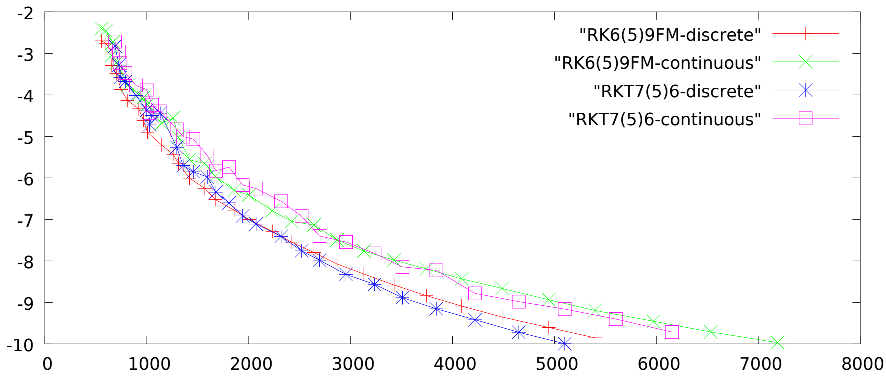


Figure 13: BRUS Problem - $log_{10}M/M^*$ against Function Evaluations.

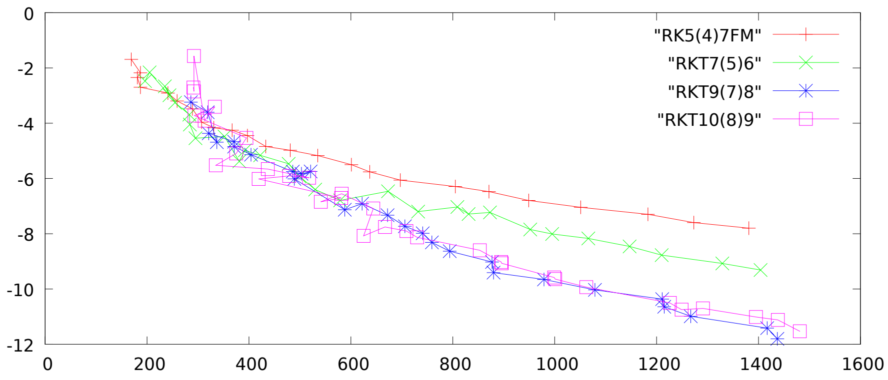


Figure 14: DETEST Problem A3 - $log_{10}M$ against Function Evaluations.

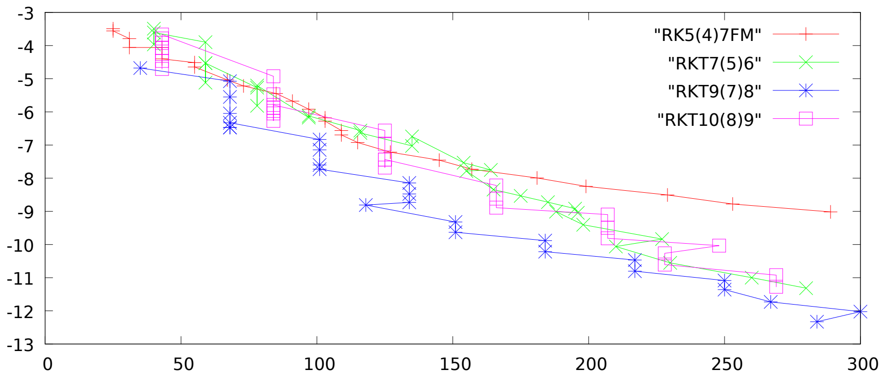


Figure 15: DETEST Problem E5 - $log_{10}M$ against Function Evaluations.

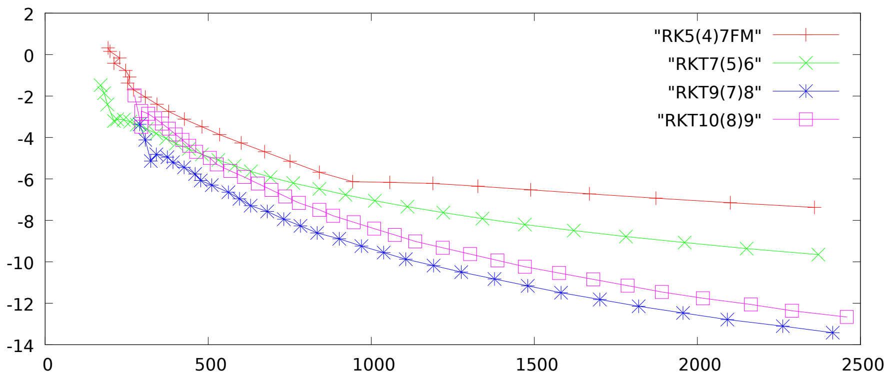


Figure 16: DETEST Problem D3 - $log_{10}M$ against Function Evaluations.

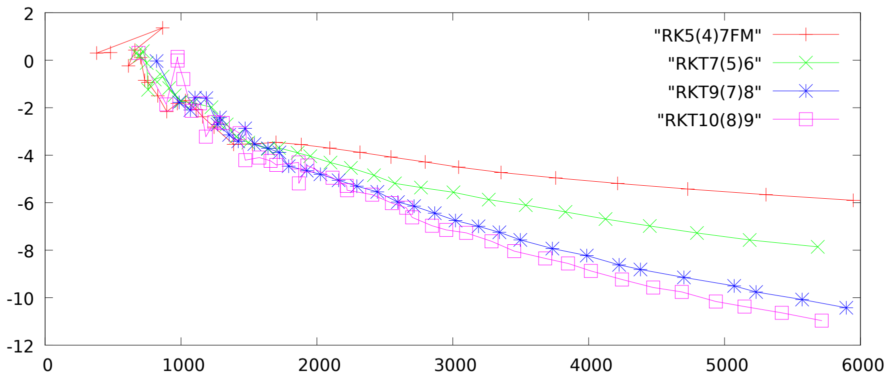


Figure 17: AREN Problem - $log_{10}M$ against Function Evaluations.

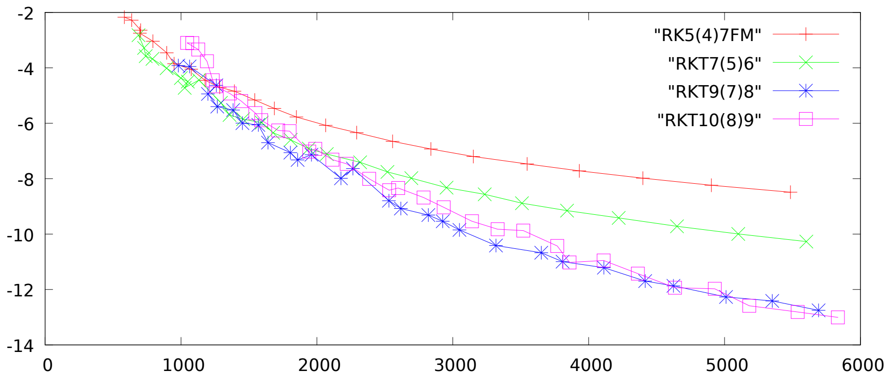


Figure 18: BRUS Problem - $log_{10}M$ against Function Evaluations.

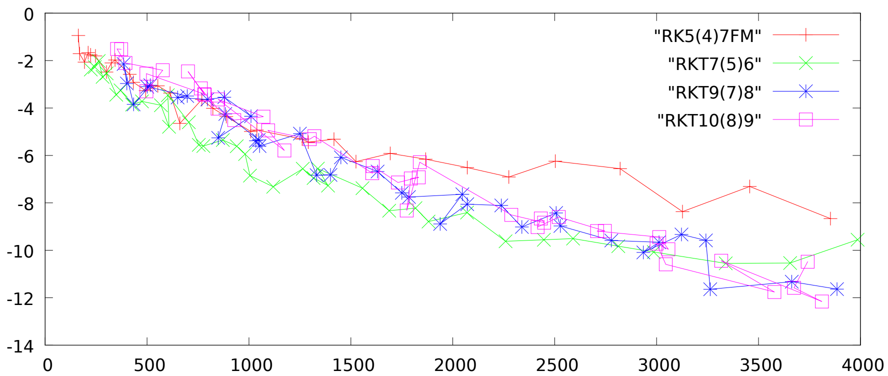


Figure 19: Modified EULR problem - $log_{10}M$ against Function Evaluations.

# 4 Summary and Discussion

This work advances the knowledge relating to the minimum number of stages necessary for an explicit Runge-Kutta process to be of a particular order in both the discrete and continuous cases (see Table 1). The suggestion made in [9] that new methods be developed which use all function evaluations at each stage to obtain both discrete and continuous approximations has been investigated for processes of order $q$, $q^* = 3, 4, ..., 9$. For each order, $q^*$, this development uses a strategy of first solving the RK continuous equations of condition, then obtaining a discrete process of order $q^*$ by enforcing the $C^1$ continuity conditions and then choosing any remaining free parameters to satisfy an efficiency condition relating global error to cost. Notice that for the higher orders this allows the discrete formula to be made order $q = q^* + 1$. Finally an embedded discrete formula of order $p$ $(= q - 1$ or $q - 2)$ is obtained which can be used for step size control.

The new triples have been extensively tested against currently available processes using standard test problems. As expected the new triples of orders 3 and 4 are generally not as efficient as those of higher order. The new triple of order 5 is found to be less efficient than the RK5(4)7FM (plus fourth order continuous process) which is therefore highly recommended when low accuracy is required. If high accuracy is required the new triples RKT9(7)8 and RKT10(8)9 are strongly recommended.

# 5 Acknowledgements

The author is grateful to his long time friend and colleague, Dr. John R. Dormand, for the many significant discussions held over the period of the development of this work. Thanks is also attributed for the use of the software packages g95, gfortran, gnuplot, latex and Maxima without which much of the work would not have been possible.

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